

# The polytopologies of transfinite provability logic

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## Abstract

Provability logics are modal or polymodal systems designed for modeling the behavior of Gödel's provability predicate in arithmetical theories and its natural extensions. If  $\Lambda$  is any ordinal, the Gödel-Löb calculus  $\text{GLP}_\Lambda$  contains one modality  $[\lambda]$  for each  $\lambda < \Lambda$ , representing provability predicates of increasing strength.  $\text{GLP}_\Lambda$  has no Kripke models, but Beklemishev and Gabelaia recently proved that  $\text{GLP}_\omega$  is complete for its class of topological models.

In this paper we generalize Beklemishev and Gabelaia's result to  $\text{GLP}_\Lambda$  for arbitrary  $\Lambda$ . We also introduce *provability ambiances*, which are topological models where valuations of formulas are restricted. With this we show completeness of  $\text{GLP}_\Lambda$  for the class of provability ambiances based on Icard polytopologies.

## 1 Introduction

Provability logic interprets modal operators as derivability predicates in order to study the behavior of formal theories. One may interpret the modal formula  $\Box\phi$  as  $\phi$  is derivable in the theory  $T$ . In [18], Solovay proved that if  $T$  is able to do a reasonable amount of arithmetic, the set of validities over the unimodal language is given by the Gödel-Löb logic  $\text{GL}$ , written  $\text{GLP}_1$  in the current paper's notation. This logic may also be interpreted over *scattered* spaces (defined later), thus giving provability a surprising connection to topology. However, in practice these semantics are somewhat heavy-handed for such a logic, which already has finite Kripke models based on transitive, well-founded frames [17].

For Japaridze's polymodal provability logic, the story is not as simple. It is an extension of  $\text{GL}$  known as  $\text{GLP}$  or, in our notation,  $\text{GLP}_\omega$  [14]. Here one considers countably many provability modalities  $[n]$ , for  $n < \omega$ . The formula  $[n]\phi$  could be interpreted (for example) as  $\phi$  is derivable using  $n$  instances of the  $\omega$ -rule. There is great interest in  $\text{GLP}$  since these logics are quite powerful and useful; Beklemishev has shown how  $\text{GLP}$  can be used to perform ordinal analysis of Peano Arithmetic and its natural subtheories [2].

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However, the logic is no longer as easy to work with as in the unimodal case. As we shall discuss later, it has no non-trivial Kripke frames. Thus the topological interpretation of the logic gives a reasonable alternative, but even then we do not get an immediate solution to the problem. In fact, the existence of so-called *canonical ordinal models* for these theories goes well beyond ZFC, as shown by Blass [7], Beklemishev [3] and in recent unpublished work by Bagaria.

There are, however, polytopologies based on ordinals for which  $\text{GLP} = \text{GLP}_\omega$  is sound and complete, as shown by Beklemishev and Gabelaia [5]. The proof of this difficult result requires some heavy machinery including Zorn's lemma. There are also simpler spaces which provide semantics for the *closed fragment*, where no free variables occur; these were introduced by Icard [11] and are closely tied to Ignatiev's Kripke model for the same fragment [13].

Our goal is to show how the constructions from [5] may be extended to the logics  $\text{GLP}_\Lambda$ , where  $\Lambda$  is an arbitrary ordinal. Here, one has transfinitely many provability operators, which as in the case of  $\text{GLP}_\omega$  represent derivability in stronger and stronger theories. Indeed, Beklemishev and Gabelaia's techniques carry over smoothly to the transfinite setting, and rather than give a new, self-contained completeness proof, we shall state the necessary results from [5] without proof in order to focus on applying these techniques beyond  $\omega$ . A key point is the computation of the higher-order rank functions, which give us upper and lower bounds on the ordinals we need in order to build models. We shall also show how the use of non-constructive topologies may be circumvented and replaced by Icard topologies by passing to a more general class of models called *ambiances*.

**Layout.** In Section 2 we give a quick overview of the logics  $\text{GLP}_\Lambda$ , and Section 3 reviews topological semantics. Section 4 then states some basic facts about ordinal arithmetic that we shall need.

Section 5 introduces the most important functions in the study of GLP-spaces, ranks and  $d$ -maps. Then, Section 6 discusses Icard ambiances and Section 7 simple ambiances, which represent the minimal structures in our framework.

After this, Section 8 discusses Beklemishev-Gabelaia spaces, which are particularly well-behaved GLP-spaces. In Section 9, we discuss and construct reductive functions, an important type of  $d$ -map, and Section 10 establishes a series of operations on ambiances which are used for constructing models.

We then go on to introduce the logic J in Section 11, which is a key ingredient in the completeness proof presented in Section 12. Finally, Section 13 uses worms to give a lower bound on the rank of models.

## 2 The logic $\text{GLP}_\Lambda$

Given any ordinal  $\Lambda$ , we can define a provability logic with modalities in  $\Lambda$ . Formulas of the language  $L_\Lambda$  are built from  $\perp$  and a countable set of propositional

variables  $\mathbb{P}$  using Boolean connectives  $\neg, \wedge$  and a modality  $[\xi]$  for each  $\xi < \Lambda$ . As is customary, we use  $\langle \xi \rangle$  as a shorthand for  $\neg[\xi]\neg$ .

The logic  $\text{GLP}_\Lambda$  is then given by the following rules and axioms:

1. all propositional tautologies,
2.  $[\xi](\phi \rightarrow \psi) \rightarrow ([\xi]\phi \rightarrow [\xi]\psi)$  for all  $\xi < \Lambda$ ,
3.  $[\xi]([\xi]\phi \rightarrow \phi) \rightarrow [\xi]\phi$  for all  $\xi < \Lambda$ ,
4.  $[\xi]\phi \rightarrow [\zeta]\phi$  for  $\xi < \zeta < \Lambda$ ,
5.  $\langle \xi \rangle \phi \rightarrow [\zeta] \langle \xi \rangle \phi$  for  $\xi < \zeta < \Lambda$ ,
6. modus ponens and
7. necessitation for each  $[\xi]$ .

Let us write  $\text{sub}(\phi)$  for the set of subformulas of  $\phi$ . Then, say  $\lambda$  *appears* in  $\phi$  if there is some formula  $\psi$  such that  $[\lambda]\psi \in \text{sub}(\phi)$ . It is evident that only finitely many ordinals may appear in any formula  $\phi$ ; sometimes it is convenient to ignore all other ordinals. To this end we define the *condensation* of  $\phi$  as follows:

**Definition 2.1.** *Given a formula  $\phi \in \text{L}_\Lambda$  such that*

$$\lambda_0 < \lambda_1 < \dots < \lambda_{I-1}$$

*are the ordinals appearing in  $\phi$ , we define a formula  $\phi^c$  (the condensation of  $\phi$ ) as the result of replacing every operator  $[\lambda_i]$  in  $\phi$  by  $[i]$ .*

As it turns out, the formula  $\phi^c$  is derivable if and only if  $\phi$  is. One direction is non-trivial and proven in [4]; the other is quite straightforward.

**Lemma 2.1.** *If  $\phi$  is a formula such that there are  $I$  ordinals appearing in  $\phi$  then  $\text{GLP}_I \vdash \phi^c$  implies that  $\text{GLP}_\Lambda \vdash \phi$ .*

This fact may be proven by uniformly substituting  $[\lambda_i]$  for  $[i]$  in a derivation of  $\phi^c$ ; we omit the details. Condensations will allow us to focus only on ‘relevant’ ordinals when analyzing formulas.

We shall also work with Kripke semantics. A *Kripke frame* is a structure  $\mathfrak{F} = \langle W, \langle R_i \rangle_{i < I} \rangle$ , where  $W$  is a set and  $\langle R_i \rangle_{i < I}$  a family of binary relations on  $W$ . A *valuation* on  $\mathfrak{F}$  is a function  $\llbracket \cdot \rrbracket : \text{L}_\Lambda \rightarrow \mathcal{P}(W)$  such that

$$\begin{aligned} \llbracket \perp \rrbracket &= \emptyset \\ \llbracket \neg \phi \rrbracket &= W \setminus \llbracket \phi \rrbracket \\ \llbracket \phi \wedge \psi \rrbracket &= \llbracket \phi \rrbracket \cap \llbracket \psi \rrbracket \\ \llbracket \langle i \rangle \phi \rrbracket &= R_i^{-1} \llbracket \phi \rrbracket. \end{aligned}$$

A *Kripke model* is a Kripke frame equipped with a valuation  $\llbracket \cdot \rrbracket$ . Note that propositional variables may be assigned arbitrary subsets of  $W$ . Often we will write  $\langle \mathfrak{F}, x \rangle \models \psi$  instead of  $x \in \llbracket \psi \rrbracket$ . As usual,  $\phi$  is *satisfied* on  $\mathfrak{F}$  if  $\llbracket \phi \rrbracket \neq \emptyset$ , and *valid* on  $\mathfrak{F}$  if  $\llbracket \phi \rrbracket = W$ .

It is well-known that polymodal GL is sound for  $\mathfrak{F}$  whenever  $R_i^{-1}$  is well-founded and transitive, in which case we write it  $<_i$ . However, constructing models of  $\text{GLP}_\Lambda$  is substantially more difficult than constructing models of GL. The full logic  $\text{GLP}_\Lambda$  cannot be sound and complete with respect to any class of Kripke frames. Indeed, let  $\mathfrak{F} = \langle W, \langle <_\xi \rangle_{\xi < \lambda} \rangle$  be a polymodal frame.

Then, it is not too hard to check that

1. Löb's axiom  $[\xi](\llbracket \xi \rrbracket \phi \rightarrow \phi) \rightarrow \llbracket \xi \rrbracket \phi$  is valid if and only if  $<_\xi$  is well-founded and transitive,
2. the axiom  $[\xi]\phi \rightarrow [\zeta]\phi$  for  $\xi \leq \zeta$  is valid if and only if, whenever  $w <_\zeta v$ , then  $w <_\xi v$ , and
3.  $\langle \xi \rangle \phi \rightarrow [\zeta] \langle \xi \rangle \phi$  for  $\xi < \zeta$  is valid if, whenever  $v <_\zeta w$ ,  $u <_\xi w$  and  $\xi < \zeta$ , then  $u <_\xi v$ .

Suppose that for  $\xi < \zeta$ , there are two worlds such that  $w <_\zeta v$ . Then from 2 we see that  $w <_\xi v$ , while from 3 this implies that  $w <_\xi w$ . But this clearly violates 1. Hence if  $\mathfrak{F} \models \text{GLP}$ , it follows that all accessibility relations (except possibly  $<_0$ ) are empty.

This observation makes the topological completeness of  $\text{GLP}_\omega$  established in [5] particularly surprising. Moreover, as we shall see, the techniques introduced there readily extend to the transfinite; but first, let us review topological semantics of provability logic.

### 3 Topological semantics

Recall that a *topological space* is a pair  $\mathfrak{X} = \langle X, \mathcal{T} \rangle$  where  $\mathcal{T} \subseteq \mathcal{P}(X)$  is a family of sets called ‘open’ such that

1.  $\emptyset, X \in \mathcal{T}$
2. if  $U, V \in \mathcal{T}$ , then  $U \cap V \in \mathcal{T}$  and
3. if  $\mathcal{U} \subseteq \mathcal{T}$  then  $\bigcup \mathcal{U} \in \mathcal{T}$ .

Given  $A \subseteq X$  and  $x \in A$ , we say  $x$  is a *limit point* of  $A$  if, for all  $U \in \mathcal{T}$  such that  $x \in U$ , we have that  $(A \setminus \{x\}) \cap U \neq \emptyset$ . We denote the set of limit points of  $A$  by  $dA$ , and call it the ‘derived set’ of  $A$ . We can define topological semantics for modal logic by interpreting Boolean operators in the usual way and setting

$$\llbracket \Diamond \psi \rrbracket_{\mathfrak{X}} = d \llbracket \psi \rrbracket_{\mathfrak{X}}.$$

In order to interpret provability logic, we will need to consider *scattered* spaces. A topological space  $\langle X, \mathcal{T} \rangle$  is scattered if every non-empty subset  $A$  of

$X$  has an isolated point; that is, if there exist  $x \in A$  and a neighborhood  $U$  of  $x$  (i.e.,  $x \in U \in \mathcal{T}$ ) such that  $U \cap A = \{x\}$ .

Many interesting examples of scattered spaces come from ordinals. The simplest is the *initial segment topology*. If  $\Theta$  is an ordinal, we write  $\Theta_0$  for the structure  $\langle \Theta, \mathcal{T} \rangle$ , where  $\mathcal{T}$  consists of all downward-closed subsets of  $\Theta$ . It is very easy to check that  $\Theta_0$  is a scattered topological space, for if  $A \subseteq \Theta$  is non-empty, then the least element of  $A$  is isolated in  $A$ .

A second important example is the *interval topology*. This is generated by all intervals on  $\Theta$  of the form  $[0, \beta]$  or  $(\alpha, \beta]$ . The interval topology extends the initial segment topology, and it is straightforward to check that if  $\mathcal{T}$  is scattered and  $\mathcal{T}'$  is any refinement of  $\mathcal{T}$  (i.e.,  $\mathcal{T} \subseteq \mathcal{T}'$ ), then  $\mathcal{T}'$  is scattered as well. We will write  $\Theta$  equipped with the interval topology as  $\Theta_1$ .

Now, in order to interpret  $\text{GLP}_\Lambda$  for  $\Lambda > 1$ , we need to consider *polytopological spaces*. A polytopological space is a structure  $\mathfrak{X} = \langle X, \langle \mathcal{T}_\lambda \rangle_{\lambda < \Lambda} \rangle$ , where  $\Lambda$  is an ordinal and each  $\mathcal{T}_\lambda$  is a topology. The derived set operator corresponding to  $\mathcal{T}_\lambda$  shall be denoted  $d_\lambda$ . We may also write  $\mathfrak{X}_\lambda$  instead of  $\langle X, \mathcal{T}_\lambda \rangle$ .

Sometimes it will also be convenient to restrict our attention to specific algebras of sets, in the spirit of generalized Kripke frames. The proper such algebras are given by the following definition:

**Definition 3.1** (*d-algebra*). *A d-algebra over a polytopological space  $\mathfrak{X} = \langle X, \langle \mathcal{T}_\lambda \rangle_{\lambda < \Lambda} \rangle$  is a collection of sets  $\mathcal{A} \subseteq \mathcal{P}(X)$  which form a Boolean algebra and such that, whenever  $\lambda < \Lambda$  and  $S \in \mathcal{A}$ , then  $d_\lambda S \in \mathcal{A}$ .*

Below we introduce ambiances, which will be the basis of our semantics; they are a slight generalization of polytopological models, which correspond to the special case where  $\mathcal{A} = \mathcal{P}(X)$ .

**Definition 3.2** (*Ambiance*). *An ambiance is a structure*

$$\mathfrak{X} = \langle X, \vec{\mathcal{T}}, \mathcal{A} \rangle$$

*consisting of a polytopological space equipped with a d-algebra  $\mathcal{A}$ .*

If  $\vec{\mathcal{T}} = \langle \mathcal{T}_\lambda \rangle_{\lambda < \Lambda}$ , we may also say  $\mathfrak{X}$  is a  $\Lambda$ -*ambiance*.

The operator  $d_\lambda$  will be used to interpret  $\langle \lambda \rangle$ :

**Definition 3.3.** *Let  $\mathfrak{X} = \langle X, \vec{\mathcal{T}}, \mathcal{A} \rangle$  be a  $\Lambda$ -ambiance.*

*A valuation on  $\mathfrak{X}$  is a function  $\llbracket \cdot \rrbracket : \mathbb{L}_\Lambda \rightarrow \mathcal{A}$  defined as in the case of Kripke semantics except that*

$$\llbracket \langle \lambda \rangle \phi \rrbracket = d_\lambda \llbracket \phi \rrbracket.$$

*A polytopological model is an ambiance equipped with a valuation.*

Let us check the conditions under which  $\text{GLP}_\Lambda$  is sound for a given polytopological space.

**Lemma 3.1.** *Let  $\Lambda$  be an ordinal and  $\mathfrak{X} = \langle X, \vec{\mathcal{T}}, \mathcal{A} \rangle$  be an ambiance. Then,*

1. Löb's axiom  $[\xi]([\xi]\phi \rightarrow \phi) \rightarrow [\xi]\phi$  is valid on  $\mathfrak{X}$  whenever  $\langle X, \mathcal{T}_\xi \rangle$  is scattered,
2. the axiom  $[\xi]\phi \rightarrow [\zeta]\phi$  for  $\xi \leq \zeta$  is valid whenever  $\mathcal{T}_\xi \subseteq \mathcal{T}_\zeta$  and
3.  $\langle \xi \rangle \phi \rightarrow [\zeta]\langle \xi \rangle \phi$  for  $\xi < \zeta$  is valid if, whenever  $A \in \mathcal{A}$ ,  $d_\xi A \in \mathcal{T}_\zeta$ .

*Proof.* See, for example, [5]. □

An ambience satisfying the above properties will be called a *provability ambience*.

When referring to topologies, we use the words *extension* and *refinement* indistinctly. We may also speak of refinements of *spaces* rather than refinements of topologies:  $\langle X', \mathcal{T}' \rangle$  is a refinement of  $\langle X, \mathcal{T} \rangle$  if  $X = X'$  and  $\mathcal{T} \subseteq \mathcal{T}'$ . Thus the condition for  $[\xi]\phi \rightarrow [\zeta]\phi$  can be rewritten as “ $\mathcal{T}_\zeta$  is a refinement of  $\mathcal{T}_\xi$ ”.

Conditions 2 and 3 suggest a very natural candidate for  $\mathcal{T}_{\xi+1}$  whenever  $\mathcal{T}_\xi$  is given; namely, the *least* topology that will satisfy all axioms.

**Definition 3.4** ( $d^A\mathcal{T}$ ). *Given a  $\mathfrak{X} = \langle X, \mathcal{T} \rangle$  and a  $d$ -algebra  $\mathcal{A}$  on  $\mathfrak{X}$ , we define  $d^A\mathcal{T}$  to be the topology on  $X$  generated by*

$$\mathcal{T} \cup \{dS : S \in \mathcal{A}\}.$$

*We will denote  $\langle X, d^A\mathcal{T} \rangle$  by  $d^A\mathfrak{X}$ .*

As in [5], we shall write  $\mathfrak{X}^+$  instead of  $d^{\mathcal{P}(X)}\mathfrak{X}$ . The above definition suggests natural candidate topologies for  $\mathcal{T}_\lambda$ , at least for successor  $\lambda$ . For limit  $\lambda$  we need to consider *directed joins* of topologies.

If  $\mathcal{T}_{\xi < \lambda}$  is an increasing sequence of topologies, then  $\bigcup_{\xi < \lambda} \mathcal{T}_\xi$  typically is not itself a topology. However, it does generate a least topology  $\mathcal{S} = \bigsqcup_{\xi < \lambda} \mathcal{T}_\xi$  containing all  $\mathcal{T}_\xi$ , in the following way: say  $U \in \mathcal{S}$  if and only if for every  $x \in U$  there is  $\xi < \lambda$  and  $V \in \mathcal{T}_\xi$  with  $x \in V$ .

Due to the monotonicity axiom, this is the least topology we can choose at limit stages:

**Definition 3.5.** *Given an ambience  $\mathfrak{X} = \langle X, \vec{\mathcal{T}}, \mathcal{A} \rangle$  and an ordinal  $\xi > 0$ , define  $\mathcal{T}_\xi^-$  to be*

- $d^A\mathcal{T}_\zeta$  if  $\xi = \zeta + 1$
- $\bigsqcup_{\zeta < \xi} \mathcal{T}_\zeta$  if  $\xi$  is a limit ordinal.

Then, say a structure  $\mathfrak{X} = \langle \Theta, \langle \mathcal{T}_\lambda \rangle_{\lambda < \Lambda} \rangle$  is a *canonical ordinal model* if  $\mathfrak{X}_0 = \Theta_1$  and for all  $\lambda < \Lambda$ ,  $\mathcal{T}_\lambda = \mathcal{T}_\lambda^-$ .

The topology  $\mathcal{T}^+$  is usually much bigger than  $\mathcal{T}$ , since we are adding many new closed sets as opens. For example:

**Lemma 3.2.** *Given an ordinal  $\Theta$ ,  $\Theta_0^+ = \Theta_1$ .*

For a proof of this fact see, e.g., [5]. After this the topologies increase very quickly; if  $\Theta$  is any countable ordinal and  $\mathcal{T}$  is the interval topology, then  $\mathcal{T}^+$  is discrete. Moreover, the question of whether  $\text{GLP}_2$  is complete for its class of canonical ordinal models is independent of ZFC [7, 3]. In recent unpublished work, Bagaria has characterized non-trivial ordinal models for  $\text{GLP}_n$  in terms of large cardinals.

Thus, making the topologies as small as possible at each step is not the best strategy, so it is convenient to consider other alternatives. In [5], Beklemishev and Gabelaia realized the highly unintuitive fact that if we make each topology *as large as possible* then subsequent topologies become much smaller! As we shall see, this idea readily extends beyond  $\text{GLP}_\omega$ ; perhaps the most technically challenging aspect of such an extension lies in the new computations with ordinals that arise.

## 4 Operations on ordinals

Before continuing, let us give a brief review of some notions of ordinal arithmetic as well as some useful functions in the study of provability logic. We skip most proofs; for further details on ordinal arithmetic, we refer the reader to a text such as [16], while the material on hyperexponentials and hyperlogarithms is treated in detail in [8].

We assume familiarity with ordinal sums, products and exponents. We shall also use the following operations:

**Lemma 4.1.**

1. Whenever  $\zeta < \xi$ , there exists a unique ordinal  $\eta$  such that  $\zeta + \eta = \xi$ . We will denote this unique  $\eta$  by  $-\zeta + \xi$ .
2. Given  $\xi > 0$ , there exist ordinals  $\alpha, \beta$  such that  $\xi = \alpha + \omega^\beta$ . The value of  $\beta$  is uniquely defined. We will denote this unique  $\beta$  by  $\ell\xi$ .

In previous work my colleague Joost Joosten and I realized that there were some particularly useful functions that arise when studying provability logics. They are *hyperexponentials* and *hyperlogarithms*, and are a form of transfinite iteration of the functions  $-1 + \omega^\xi$  and  $\ell$ , respectively. These iterations have been used in [10] for describing well-orders in the Japardize algebra and in [9] for defining models of the variable-free fragment of  $\text{GLP}_\Lambda$ . They will be essential in defining our semantics. We give only a very brief overview, but [8] gives a thorough and detailed presentation.

We shall denote the class of all ordinals by  $\text{On}$  and the class of limit ordinals by  $\text{Lim}$ .

**Definition 4.1.** Let  $e(\xi) = -1 + \omega^\xi$ . Then, we define the hyperexponentials  $\langle e^\zeta \rangle_{\zeta \in \text{On}}$  as the unique family of normal<sup>1</sup> functions such that

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<sup>1</sup>That is, strictly increasing and continuous.

1.  $e^1 = e$
2.  $e^{\alpha+\beta} = e^\alpha e^\beta$  for all ordinals  $\alpha, \beta$
3.  $\langle e^\xi \rangle_{\xi \in \text{On}}$  is pointwise minimal amongst all families of normal functions satisfying the above clauses<sup>2</sup>.

It is not obvious that such a family of functions exists, but a detailed construction is given in [8], where the following is also proven:

**Proposition 4.1** (Properties of hyperexponentials). *The family of functions  $\langle e^\xi \rangle_{\xi \in \text{On}}$  has the following properties:*

1. given  $\xi \in \text{On}$ ,  $e^{\xi+1} = \lim_{n \rightarrow \omega} e^\xi n$
2. given  $\xi \in \text{Lim}$ ,  $e^\xi 1 = \lim_{\zeta \rightarrow \xi} e^\zeta 1$
3. if  $\lambda \in \text{Lim}$  and  $\vartheta \in \text{On}$ ,  $e^\lambda(\vartheta + 1) = \lim_{\eta \rightarrow \lambda} e^\eta(e^\lambda(\vartheta) + 1)$ .

Closely related to hyperexponentials are hyperlogarithms. Below, an *initial function* is one mapping initial segments to initial segments.

**Definition 4.2** (Hyperlogarithms). *We define the sequence  $\langle \ell^\xi \rangle_{\xi \in \text{On}}$  to be the unique family of initial functions such that*

1.  $\ell^1 = \ell$ ,
2.  $\ell^{\alpha+\beta} = \ell^\beta \ell^\alpha$  for all ordinals  $\alpha, \beta$ ,
3.  $\langle \ell^\xi \rangle_{\xi \in \text{On}}$  is pointwise maximal among all families of functions satisfying the above clauses.

Hyperexponentials are not surjective, hence not invertible. However, they do admit left inverses, and these are provided by hyperlogarithms:

**Lemma 4.1.** *If  $\xi < \zeta$ , then  $\ell^\xi e^\zeta = e^{-\xi+\zeta}$ .*

*Further, whenever  $\alpha < e^\xi \beta$ , it follows that  $\ell^\xi \alpha < \beta$ .*

There is a close relation between the iterates  $e^{\omega^\rho} \xi$  and Veblen functions; this is also described in detail in [8]. For example:

**Lemma 4.2.** *An ordinal  $\xi$  lies in the range of  $e^{\omega^\gamma}$  if and only if, for all  $\delta < \gamma$ , we have that  $\xi = e^{\omega^\delta} \xi$ . In particular,  $e^{\omega^{\gamma+1}}$  enumerates the fixpoints of  $e^{\omega^\gamma}$ .*

As a direct consequence we get the following:

**Lemma 4.3.** *Suppose that for ordinals  $\vartheta, \gamma$  and additively indecomposable  $\Lambda$  we have that  $\vartheta \in (e^\Lambda \gamma, e^\Lambda(\gamma + 1))$ .*

*Then, there exists  $\lambda < \Lambda$  such that  $\ell^\lambda \vartheta \leq e^\Lambda \gamma$ .*

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<sup>2</sup>That is, if  $\langle g^\xi \rangle_{\xi \in \text{On}}$  is a family of functions satisfying conditions 1 and 2, then for all ordinals  $\xi, \zeta$ ,  $e^\zeta \xi \leq g^\zeta \xi$ .



*Proof.* The sequence  $\langle \ell^\lambda \vartheta \rangle_{\lambda < \Lambda}$  is non-increasing and hence reaches a minimum value at some  $\lambda^* < \Lambda$ ; it is easy to see then that  $\ell^\delta \ell^{\lambda^*} \vartheta = \ell^{\lambda^*} \vartheta$  for all  $\delta < \Lambda$ . It then follows that  $e^\delta \ell^{\lambda^*} \vartheta = \ell^{\lambda^*} \vartheta$ , for otherwise we would have that  $e^\delta \ell^{\lambda^*} \vartheta > \ell^{\lambda^*} \vartheta$  since  $e^\delta$  is normal and thus by Lemma 4.1,  $\ell^\delta \ell^{\lambda^*} \vartheta < \ell^{\lambda^*} \vartheta$ , which cannot be.

It follows by Lemma 4.2 that  $\ell^{\lambda^*} \vartheta = e^\Lambda \gamma'$  for some  $\gamma'$ , and since  $e^\Lambda$  is increasing we must have  $e^\Lambda \gamma' \leq e^\Lambda \gamma$ .  $\square$

To conclude this section, let us discuss simple functions.

**Definition 4.3.** A simple function is a function  $s : \Lambda \rightarrow \Theta$  such that  $s(\xi) = 0$  for all but finitely many values of  $\xi$ .

We will define the *support* of  $r$  to be the set of ordinals on which it is non-zero and denote it  $\text{supp}(r)$ . If  $r, s$  are simple functions, we define  $r \sqcup s$  to be the simple function given by  $r \sqcup s(\lambda) = \max\{r(\lambda), s(\lambda)\}$ .

We write  $s \sqsubseteq \alpha$  if for all  $\lambda < \Lambda$ ,  $s(\lambda) \leq \ell^\lambda \alpha$ . Given a simple function  $s$ , we define  $\lceil s \rceil$  to be the least ordinal  $\xi$  such that  $s \sqsubseteq \xi$ .

The following is a mild variant of a claim proven in [9]:

**Lemma 4.4.** Let  $r$  be a simple function and  $\lambda = \max(\text{supp}(r))$ .

Then, the ordinal  $\lceil r \rceil$  is defined and  $\ell^\lambda \lceil r \rceil = r(\lambda)$ .

Further, whenever  $r \sqsubseteq \alpha$  and  $\xi < \lambda$ , we have that  $\ell^\xi \lceil r \rceil \leq \ell^\xi \alpha$ .

## 5 Ranks and $d$ -maps

In this section we shall consider some important basic concepts in the study of scattered spaces. We omit the proofs of those results which may already be found in [5].

Given a topological space  $\langle X, \mathcal{T} \rangle$ , we may iterate the corresponding derived set operator  $d : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$  via the following recursion:

1.  $d^0 A = A$
2.  $d^{\xi+1} A = dd^\xi A$  for all  $\xi \in \text{On}$
3.  $d^\lambda A = \bigcap_{\xi < \lambda} d^\xi A$  for  $\lambda \in \text{Lim}$ .

If  $X$  is scattered and  $d^\xi A \neq \emptyset$ , then  $d^\xi A$  contains an isolated point  $x$  and thus  $x \notin dd^\xi A = d^{\xi+1} A$ . In particular,  $d^\xi X \supsetneq d^{\xi+1} X$  provided  $d^\xi X \neq \emptyset$ .

Thus if  $\# \xi > \# X$ ,  $d^\xi X = \emptyset$ , which means that for any point  $x \in X$  there is some ordinal such that  $x \notin d^\xi X$ . This motivates our following definition:

**Definition 5.1 (Rank).** If  $\mathfrak{X} = \langle X, \mathcal{T} \rangle$  is a scattered space and  $x \in X$ , we define the rank of  $x$ , denoted  $\rho(x)$ , to be the least ordinal  $\alpha$  such that  $x \notin d^{\alpha+1} X$ .

We define  $\rho(\mathfrak{X}) = \sup_{x \in X} (\rho(x) + 1)$ . This is the Cantor-Bendixon rank of  $\mathfrak{X}$ .

Let us mention some useful equalities established in [5]:

**Lemma 5.1.** *Given an ordinal  $\Theta$ , let  $\rho_0$  be the rank function on  $\Theta_0$  and  $\rho_1$  the rank function on  $\Theta_1$ .*

*Then, for all  $\xi < \Theta$ ,  $\rho_0(\xi) = \xi$  while  $\rho_1(\xi) = \ell\xi$ .*

Of course, the definitions we have given do not immediately tell us how to compute ranks. The following is an important tool for doing so:

**Lemma 5.2.** *Given a scattered space  $\langle X, \mathcal{T} \rangle$  with rank function  $\rho$  and  $x \in X$ , we have that if  $V$  is any neighborhood of  $x$ , then  $\rho(V \setminus \{x\}) \supseteq [0, \rho(x))$ .*

*Moreover, there is a neighborhood  $U$  of  $x$  with  $\rho(U \setminus \{x\}) = [0, \rho(x))$ .*

As the derived set operator is central to the semantics of  $\text{GLP}_\Lambda$ , we need to focus on those operators that preserve it. Of course,  $d$  is homeomorphism-invariant, but this class of maps is too restrictive. Meanwhile, if  $f$  is merely continuous and open, it is not generally the case that  $df = d!$ . As a simple counterexample, consider the ordinals 1 and  $\omega + 1$  equipped with the interval topology, and let  $f : \omega + 1 \rightarrow 1$  be the map that is identically zero. Of course, this is the only function between the two spaces. Further, it is easily checked to be continuous and open, yet  $d(\omega + 1) = \{\omega\}$  while  $d1 = \emptyset$ .

To this end, we need to consider  $d$ -maps. Recall that a space is *discrete* if every subset is open.

**Definition 5.2** ( $d$ -map). *Given topological spaces  $\mathfrak{X} = \langle X, \mathcal{T} \rangle$  and  $\mathfrak{Y} = \langle Y, \mathcal{S} \rangle$ , a  $d$ -map from  $\mathfrak{X}$  to  $\mathfrak{Y}$  is a function<sup>3</sup>  $f : \mathfrak{X} \rightarrow \mathfrak{Y}$  which is continuous, open, and such that  $f^{-1}\{y\}$  is discrete for every  $y \in Y$ .*

The latter condition is equivalent to the apparently stronger condition that  $f^{-1}A$  is discrete whenever  $A$  is. With this observation one readily obtains the following:

**Lemma 5.3.** *The composition of  $d$ -maps is a  $d$ -map.*

A useful observation about  $d$ -maps is that they preserve rank; in other words, if  $\mathfrak{X}, \mathfrak{Y}$  are scattered spaces with rank-function  $\rho_{\mathfrak{X}}, \rho_{\mathfrak{Y}}$  and  $f : \mathfrak{X} \rightarrow \mathfrak{Y}$  is a  $d$ -map, then  $\rho_{\mathfrak{X}} = \rho_{\mathfrak{Y}} f$ . We shall call functions with the latter property *rank-preserving* (even if they are not  $d$ -maps).

In fact, the rank function itself is an example of a  $d$ -map. The following is also proven in [5]:

**Lemma 5.4.** *Given a scattered space  $\mathfrak{X}$ , the function  $\rho : \mathfrak{X} \rightarrow \rho(\mathfrak{X})_0$  is a  $d$ -map. Moreover, if  $f : \mathfrak{X} \rightarrow \Theta_0$  is a  $d$ -map, it follows that  $f = \rho$ .*

Thus  $\Theta_0$  is an initial object in the category of scattered spaces with Cantor-Bendixon rank at most  $\Theta$ .

The next result shows that it is particularly easy to compute the limit points of rank-invariant sets:

**Lemma 5.5.** *If  $\mathfrak{X}$  is a scattered space with rank function  $\rho$  and  $S$  a set of ordinals then  $d\rho^{-1}S$  is the set of all  $x \in X$  such that  $\rho x > \min S$ .*

---

<sup>3</sup>We write  $f : \mathfrak{X} \rightarrow \mathfrak{Y}$  instead of  $f : X \rightarrow Y$  when the specific topologies are relevant.

*Proof.* This is a direct consequence of Lemma 5.2. Indeed, if  $\rho x \leq \min S$  then there is a neighborhood  $U$  of  $x$  such that  $\rho(U \setminus \{x\}) = [0, \rho(x))$ , and hence  $x \notin d\rho^{-1}S$ ; meanwhile, if  $\rho(x) \geq \min S$ , then given a neighborhood  $U$  of  $x$  we have by Lemma 5.4 that  $\rho(U \setminus \{x\})$  is an initial segment, and again by Lemma 5.2 it must contain  $\min S$ .  $\square$

There is one more extension of a scattered topology that it will be useful to consider. Given a topological space  $\mathfrak{X} = \langle X, \mathcal{T} \rangle$ , let  $\vec{\mathcal{T}}$  be the topology generated by  $\mathcal{T}$  and all sets of the form  $d^{\xi+1}X$  such that  $\xi \in \mathbf{On}$ .

The following claim is then proven in [5]:

**Lemma 5.6.** *If  $\mathfrak{X} = \langle X, \mathcal{T} \rangle$  is limit-maximal and  $f : \mathfrak{X} \rightarrow \mathfrak{Y}$  is a  $d$ -map then  $f : \mathfrak{X} \rightarrow \mathfrak{Y}$  is also a  $d$ -map.*

This result will be useful in extending constructions to successor modalities. We will also need the following lemma in order to deal with limit modalities:

**Lemma 5.7.** *Let  $\mathfrak{X} = \langle X, \vec{\mathcal{T}} \rangle$  and  $\mathfrak{Y} = \langle Y, \vec{\mathcal{S}} \rangle$  be  $\lambda$ -polytopologies such that both  $\vec{\mathcal{T}}$  and  $\vec{\mathcal{S}}$  are increasing.*

*If  $\lambda \in \mathbf{Lim}$  and  $f : \mathfrak{X}_\xi \rightarrow \mathfrak{Y}_\xi$  is a  $d$ -map for all  $\xi < \lambda$  then*

$$f : \left\langle X, \bigsqcup_{\xi < \lambda} \mathcal{T}_\xi \right\rangle \rightarrow \left\langle Y, \bigsqcup_{\xi < \lambda} \mathcal{S}_\xi \right\rangle$$

*is a  $d$ -map.*

*Proof.* Suppose that  $U \subseteq X$  is open and let  $y \in f(U)$ , so that for some  $x \in U$ ,  $y = f(x)$ . Then  $U$  contains a  $\xi$ -neighborhood  $V$  of  $x$  for some  $\xi < \lambda$  and thus  $f(V)$  is  $\xi$ -open. But this means that  $f(V)$  is a  $\lambda$ -neighborhood of  $y$ , and we conclude that  $f(U)$  is open.

Continuity follows from a similar reason, and given  $y \in Y$  we see that  $f^{-1}(y)$  is discrete because it is already 0-discrete<sup>4</sup>.  $\square$

## 6 Icard ambiances

In this section we shall discuss *Icard topologies*, originally introduced in [11] for  $\text{GLP}_\omega$  and generalized to arbitrary  $\text{GLP}_\Lambda$  in [9].

Let  $\mathcal{I}_0$  be the initial segment topology, and for  $0 < \lambda < \Lambda$  define a topology  $\mathcal{I}_\lambda$  on  $\Theta$  by setting, for  $\lambda < \Lambda$ ,  $\mathcal{I}_\lambda$  to be the topology generated by sets of the form

$$(\alpha, \beta]_\xi = \{\vartheta : \alpha < \ell^\xi \vartheta \leq \beta\}$$

or of the form

$$[0, \beta]_\xi = \{\vartheta : \ell^\xi \vartheta \leq \beta\}$$

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<sup>4</sup>Topological properties in polytopological spaces will be indexed according to the topology being referred to. For example, 0-discrete means *discrete in  $\mathcal{T}_0$* .

for some  $\alpha < \beta \leq \Theta$  and  $\xi < \lambda$ . For uniformity, we may write  $[0, \beta]_\xi = (-1, \beta]_\xi$ , and thus we may assume all intervals to be open on the left.

We will call the resulting polytopological space  $\mathfrak{Ic}_\Lambda^\Theta$ . We will denote the derived-set operator with respect to  $\mathcal{I}_\lambda$  by  $i_\lambda$  and the ordinal  $\Theta$  equipped with  $\mathcal{I}_\lambda$  by  $\Theta_\lambda$ ; note that there is no clash in notation in the cases  $\lambda = 0, 1$  as the Icard topologies coincide with the initial segment and interval topologies, respectively.

If  $r$  is a simple function and  $\alpha$  an ordinal, write  $r \sqsubset \alpha$  if  $r(\xi) < \ell^\xi \alpha$  for all  $\xi \in \text{supp}(r)$ . If  $r$  is a simple function such that  $r \sqsubset \alpha$ , then we can identify  $r$  with an Icard-neighborhood of  $\alpha$ . Namely, we set

$$B_r(\alpha) = \bigcap_{\xi \in \text{supp}(r)} (r(\xi), \ell^\xi \alpha]_\xi;$$

this representation will be convenient to keep in mind.

It is well-known that Icard spaces are not models of GLP, but as we shall see Icard *ambiances* are:

**Definition 6.1** (Icard ambiance). *An Icard ambiance is an ambiance  $\mathfrak{X}$  based on an Icard space.*

Icard ambiances are rather nice to work with, since the topologies are all easy to describe. We will also use *shifted* Icard ambiances, based on  $\langle \mathcal{I}_{1+\lambda} \rangle_{\lambda < \Lambda}$ ; these are important as  $\text{GLP}_1$  is incomplete for the interval topology. We will denote the shifted  $\Lambda$ -Icard space on  $\Theta$  by  $\widehat{\mathfrak{Ic}}_\Lambda^\Theta$ .

The following property will prove to be useful:

**Lemma 6.1.** *Given  $\xi \leq \Theta$  and  $\lambda < \Lambda$ , there is a  $\mathcal{T}_\lambda$ -neighborhood  $U$  of  $\xi$  such that whenever  $\xi \neq \zeta \in U$ ,  $\ell^\lambda \zeta < \ell^\lambda \xi$ .*

*Proof.* This is a slight modification of a result proven in [9].  $\square$

Constructing  $d$ -maps between Icard spaces will be crucial. Fortunately, hyperlogarithms already provide important examples:

**Lemma 6.2.** *If  $\Theta, \xi, \zeta$  are ordinals, then  $\ell^\xi : \Theta_{\xi+\zeta} \rightarrow \Theta_\zeta$  is a  $d$ -map.*

*Proof.* That  $\ell^\xi$  is pointwise discrete is an immediate consequence of Lemma 6.1.

Let us first consider the case when  $\zeta = 0$ . Let  $[0, \beta]_0$  be a 0-open set and  $\vartheta \in (\ell^\xi)^{-1}[0, \beta]_0$ . Once again use Lemma 6.1 to find a  $\xi$ -neighborhood  $U$  of  $\vartheta$  such that for all  $\eta \in U$ ,  $\ell^\xi \eta \leq \beta$ . But then,  $U \subseteq (\ell^\xi)^{-1}[0, \beta]_0$ , and since all parameters were arbitrary we conclude that  $\ell^\xi : \Theta_\xi \rightarrow \Theta_0$  is continuous.

To see that it is open, let  $U = \bigcap_{i < I} (\alpha_i, \beta_i]_{\delta_i}$  be  $\xi$ -open and suppose that  $\vartheta \in U$ . We claim that  $[0, \ell^\xi \vartheta] \subseteq \ell^\xi U$ .

To see this, pick  $\eta \leq \ell^\xi \vartheta$  and define a simple function  $r$  with  $r(\delta_i) = \alpha_i + 1$  and  $r(\xi) = \eta$ . Then, by Lemma 4.4,  $\ell^\xi[r] = \eta$  while for all  $i < I$ ,

$$\alpha_i < \ell^{\delta_i}[r] \leq \ell^{\delta_i} \vartheta \leq \beta_i.$$

Thus  $[r] \in U$ , so that  $\eta \in \ell^\xi U$ . Since  $\delta$  was arbitrary, we conclude that  $[0, \ell^\xi \vartheta]_0 \in \ell^\xi U$ , and thus  $\ell^\xi : \Theta_\xi \rightarrow \Theta_0$  is open, as claimed.

Now we must consider  $\zeta > 1$ . To see that  $\ell^\xi : \Theta_{\xi+\zeta} \rightarrow \Theta_\xi$  is continuous, note that if  $\delta < \zeta$ ,  $\ell^\xi \gamma \in (\alpha, \beta)_\delta$  if and only if  $\ell^\delta \ell^\xi \gamma = \ell^{\xi+\delta} \gamma \in (\alpha, \beta)$ , that is,  $(\ell^\xi)^{-1}(\alpha, \beta)_\delta = (\alpha, \beta)_{\xi+\delta}$ , which is  $(\xi + \zeta)$ -open.

Next, let us check that it is open. Suppose that  $\gamma \in U = \bigcap_{i < I} (\alpha_i, \beta)_{\delta_i}$  where  $\delta_i < \delta_{i+1} < \xi + \zeta$  and suppose that  $J \leq I$  is the largest index such that  $\delta_i < \xi$  for all  $i < J$ . Consider the  $\zeta$ -neighborhood

$$V = [0, \ell^\xi \gamma]_0 \cap \bigcap_{J \leq i < I} (\alpha_i, \ell^{\delta_i} \gamma]_{-\xi+\delta_i}$$

of  $\ell^\xi \gamma$ .

Now, suppose that  $\eta \in V$ . Consider  $s$  given by  $s(\delta_i) = \alpha_i + 1$  for  $i < J$  and  $s(\xi) = \eta$ , otherwise  $s \equiv 0$ . By Lemma 4.4 we know that  $\alpha_i < \lceil s \rceil \leq \ell^{\delta_i} \gamma$  for all  $i < J$ , while  $\ell^\xi \lceil s \rceil = \eta$  and for  $i \geq J$ ,  $\ell^{\delta_i} \lceil s \rceil = \ell^{-\xi+\delta_i} \eta \in (\alpha_i, \beta_i]$ . It follows that  $\lceil s \rceil \in U$  and  $\ell^\xi \lceil s \rceil = \eta$ , so that  $\eta \in \ell^\xi U$ , as claimed.  $\square$

An important corollary of this is the following:

**Corollary 6.1.** *If  $\rho_\xi$  denotes the rank with respect to  $\mathcal{I}_\xi$ , then  $\rho_\xi = \ell^\xi$ .*

*Proof.* Immediate from Lemma 6.2 with  $\zeta = 0$  and Lemma 5.4.  $\square$

With this we may also obtain another useful characterization of Icard topologies, originally due to Beklemishev:

**Lemma 6.3.** *Given ordinals  $\Theta, \xi$ ,*

1. *If  $\xi = \zeta + 1$  then  $\mathcal{I}_\xi = \dot{\mathcal{I}}_\zeta$ ,*
2. *if  $\xi \in \text{Lim}$  then  $\mathcal{I}_\xi = \bigsqcup_{\zeta < \xi} \mathcal{I}_\zeta$ .*

*Proof.* The second claim is immediate from the definitions, so we shall check only the first.

Here we note that  $\mathcal{I}_\xi$  is obtained from  $\mathcal{I}_\zeta$  by adding sets of the form  $(\alpha, \beta]_\xi$  as opens. In view of Lemma 6.1 we know that  $[0, \beta]_\xi$  is always  $\zeta$ -open, so it suffices to prove that  $(\alpha, \Theta)_\zeta$  is  $\dot{\mathcal{I}}_\zeta$ -open as well. But this follows from Theorem 6.1, as for all  $\delta$  we have that  $\rho_\zeta \delta = \ell^\zeta \delta$  and hence  $\delta \in [\alpha, \Theta)_\zeta$  if and only if  $\rho_\zeta \delta \geq \alpha$ , i.e. if  $\delta \in i_\zeta^\alpha[0, \Theta)$ .

We conclude that  $(\alpha, \Theta)_\zeta = i_\zeta^{\alpha+1}[0, \Theta)$ , which by definition is an element of  $\dot{\mathcal{I}}_\zeta$ .  $\square$

This gives us one further result:

**Lemma 6.4.** *If  $f : \Theta_\alpha \rightarrow \Xi_\beta$  is a  $d$ -map then for all  $\gamma$ ,  $f : \Theta_{\alpha+\gamma} \rightarrow \Xi_{\beta+\gamma}$  is a  $d$ -map.*

*Proof.* By a simple induction on  $\gamma$ . For successor  $\gamma$  we use Lemma 6.3 together with Lemma 5.6; for limit  $\gamma$  we use Lemma 5.7.  $\square$

## 7 The simple ambiance

It will be convenient to focus on a specific ambiance to get a feel for how these may be constructed. The simple ambiance we shall present here is not entirely central to our completeness proof since indeed  $\text{GLP}$  is not complete for simple ambiances, but simple sets nevertheless provide the appropriate semantics for the *closed fragment*  $\text{GLP}_\Lambda^0$  of  $\text{GLP}_\Lambda$ , where propositional variables may not occur (only  $\perp$ ).

It turns out that a set is simple if and only if it is definable by a closed formula; we will prove one implication later. With this equivalence in mind, the set of valid formulas of  $\mathbb{L}_\Lambda$  over the class of simple ambiances is equal to the set of validities over the closed-fragment definable sets. This logic is described in [12, 15] and extends  $\text{GLP}_\Lambda$  by the axioms for linear frames.

**Definition 7.1.** A set  $S \subseteq \Theta$  is simple if there exist finite sets  $I, J$  and ordinals  $\alpha_{ij}, \beta_{ij}, \sigma_{ij}$  such that

$$S = \bigcup_{i \in I} \bigcap_{j \in J} (\alpha_{ij}, \beta_{ij}]_{\sigma_{ij}}.$$

If all  $\sigma_{ij} \leq \lambda$ , we say  $S$  is  $\lambda$ -simple.

It is an easy observation that all  $\lambda$ -simple sets are  $\lambda$ -open.

**Lemma 7.1.** If  $S, T$  are simple sets, then  $\Theta \setminus S$ ,  $S \cap T$ ,  $S \cup T$  and  $d_\lambda S$  are simple sets; further,  $d_\lambda S$  is  $(\lambda + 1)$ -open.

*Proof.* We focus on showing that  $d_\lambda S$  is  $(\lambda + 1)$ -simple as the other properties use standard Boolean algebra manipulations.

Note that

$$d_\lambda \bigcup_{i \in I} \bigcap_{j \in J} (\alpha_{ij}, \beta_{ij}]_{\sigma_{ij}} = \bigcup_{i \in I} d_\lambda \bigcap_{j \in J} (\alpha_{ij}, \beta_{ij}]_{\sigma_{ij}},$$

so our claim will be established if we prove that  $d_\lambda \bigcap_{i < I} (\alpha_i, \beta_i]_{\sigma_i}$  is always  $(\lambda + 1)$ -simple.

Thus we suppose that

$$S = \bigcap_{i \in I} (\alpha_i, \beta_i]_{\sigma_i}.$$

Assume that  $S \neq \emptyset$ , since otherwise the claim is trivial given that  $d_\lambda \emptyset = \emptyset$ , which is  $(\lambda + 1)$ -simple, and let  $\delta \in S$ . Assume also that the  $\sigma_i$ 's are in increasing order and let  $H$  be the largest index such that  $\sigma_H < \lambda$ . Let  $r$  be a simple function defined by  $r(\sigma_i) = \alpha_i + 1$  for all  $i \in I$ ,  $r(\xi) = 0$  elsewhere, and  $\alpha_* = \ell^\lambda[r]$ .

We claim that

$$d_\lambda \bigcap_{i \in I} (\alpha_i, \beta_i]_{\sigma_i} = (\alpha_*, \Theta]_\lambda \cap \bigcap_{i \leq H} (\alpha_i, \beta_i]_{\sigma_i}.$$

Let us begin by showing that the right-hand side is contained in the left. Let  $\xi \in (\alpha_*, \Theta]_\lambda \cap \bigcap_{i \leq H} (\alpha_i, \beta_i]_{\sigma_i}$  and pick any basic  $\lambda$ -neighborhood  $B_t(\xi)$  of  $\xi$ . Let

$r'$  be a simple function which is equal to  $r$  on all  $\xi < \lambda$ , but  $r'(\lambda) = \alpha_*$  and zero elsewhere. Then define  $s = t \sqcup r'$ ; we claim that  $\zeta = \lceil s \rceil \in B_t(\xi) \cap S$ .

First note that, by Lemma 4.4,  $\zeta \in B_t(\xi)$ . Further, also using Lemma 4.4,

$$\alpha_i < \ell^{\sigma_i} \zeta \leq \ell^{\sigma_i} \xi \leq \beta_i$$

for all  $i \leq H$ , and  $\ell^\lambda \zeta = \alpha_*$  so that for  $i > H$  we see that

$$\alpha_i < \ell^{\sigma_i} \lceil r \rceil = \ell^{\sigma_i} \zeta \leq \ell^{\sigma_i} \delta \leq \beta_i,$$

and  $\zeta \in S$ .

Finally, note that  $\ell^\lambda \zeta = \alpha_* < \ell^\lambda \xi$ , and therefore  $\zeta \neq \xi$ . Since  $B_t(\xi)$  was arbitrary, we conclude that  $\xi \in d_\lambda S$ .

Now let us show that the left-hand side is contained in the right-hand side. To do this, pick  $\xi \in d_\lambda S$ . For each  $i \leq H$ , it is easy to see that  $\xi \in (\alpha_i, \beta_i]_{\sigma_i}$ ; otherwise,  $[0, \alpha_i]_{\sigma_i} \cup (\beta_i, \Theta)_{\sigma_i}$  is a neighborhood of  $\xi$  which does not intersect  $S$ . Meanwhile, if  $\ell^\lambda \xi \leq \alpha_*$ , by Lemma 6.1, there is a  $\lambda$ -neighborhood  $V$  of  $\xi$  such that if  $\zeta \neq \xi$  is contained in  $V$ , then  $\ell^\lambda \zeta < \ell^\lambda \xi$ . But by Lemma 4.4, if  $r \sqsubseteq \zeta$  then  $\ell^\lambda \zeta \geq \alpha_*$ , which means that  $V \cap (S \setminus \{\xi\}) = \emptyset$ .  $\square$

With this we may prove the following:

**Theorem 7.1.** *GLP $_\Lambda$  is sound for the class of simple ambiances.*

*Proof.* As we have seen, Icard ambiances satisfy all axioms of GLP except possibly for  $\langle \xi \rangle \phi \rightarrow [\xi + 1] \langle \xi \rangle \phi$ , but this is satisfied in any simple ambiance due to Lemma 7.1.  $\square$

We conclude by mentioning a result relating simple sets to closed formulas. The converse claim is also true, i.e. that every simple set may be defined by a closed formula, but we shall not go into details here.

**Lemma 7.2.** *Given a closed formula  $\phi$  and an Icard ambiance  $\mathfrak{X}$  with valuation  $\llbracket \cdot \rrbracket$ ,  $\llbracket \phi \rrbracket$  is a simple set.*

*Proof.* By induction on the build of  $\phi$  using Lemma 7.1 and the fact that  $\llbracket \perp \rrbracket = \emptyset$  is simple.  $\square$

## 8 Beklemishev-Gabelaia spaces

One key observation when constructing GLP-spaces is that the operation  $\cdot^+$  is not monotone. Thus it is possible that  $\mathcal{T}^+$  is discrete yet for some suitable extension  $\mathcal{T}'$  of  $\mathcal{T}$ ,  $(\mathcal{T}')^+$  is not. In this case, it will be useful to pass to such an extension. This idea is central to the completeness proof of [5].

‘Suitable’ refinements are as described in the following definition:

**Definition 8.1** (rank-preserving, limit-maximal refinement). *Let  $\mathcal{T} \subseteq \mathcal{T}'$  be two topologies on a set  $X$ . Assume  $\langle X, \mathcal{T} \rangle$  is scattered (so that  $\langle X, \mathcal{T}' \rangle$  is scattered as well). Let  $\rho, \rho'$  be the respective rank functions.*

*Then,  $\mathcal{T}'$  is a*

1. rank-preserving refinement of  $\mathcal{T}$  if  $\rho = \rho'$ ;
2. limit-refinement of  $\mathcal{T}$  if it is a rank-preserving refinement and, whenever  $\rho(\xi) \notin \text{Lim}$  and  $U$  is any  $\mathcal{T}'$ -neighborhood of  $\xi$ , there is a  $\mathcal{T}$ -neighborhood  $V$  of  $\xi$  such that  $V \subseteq U$ ;
3. limit-maximal refinement of  $\mathcal{T}$  if there is no limit-extension  $\mathcal{T}''$  of  $\mathcal{T}$  such that  $\mathcal{T}' \subsetneq \mathcal{T}''$ .

Limit-maximal refinements are very useful for constructing GLP-spaces. The following two lemmas are proven in [5] and are crucial in the construction:

**Lemma 8.1.** *If  $\mathfrak{X}$  is limit-maximal then  $\mathfrak{X}^+ = \dot{\mathfrak{X}}$ .*

**Lemma 8.2.** *If  $f : \mathfrak{X} \rightarrow \mathfrak{Y}$  is a d-map and  $\mathfrak{Y}$  is limit-maximal then there is a limit-maximal refinement  $\mathfrak{X}'$  of  $\mathfrak{X}$  such that  $f : \mathfrak{X}' \rightarrow \mathfrak{Y}$  is also a d-map.*

With this we have all the tools we need to construct Beklemishev-Gabelaia spaces. Below, we use  $\mathcal{T}_\xi^-$  in the sense of Definition 3.5.

**Definition 8.2.** *Let  $\Theta$  be an ordinal.*

*A polytopology  $\langle \mathcal{T}_\xi \rangle_{\xi < \Lambda}$  on  $\Theta$  is a Beklemishev-Gabelaia topology (BG-topology) if  $\mathcal{T}_0$  is a limit-maximal extension of the interval topology and for all  $\xi$ ,  $\mathcal{T}_\xi$  is a limit-maximal extension of  $\mathcal{T}_\xi^-$ .*

There is a close relationship between BG and Icard spaces:

**Lemma 8.3.** *If  $\mathfrak{X}$  is a  $\Lambda$ -BG space with topologies  $\vec{\mathcal{T}}$  then for all  $\lambda < \Lambda$ ,  $\mathcal{T}_\lambda$  is a rank-preserving extension of  $\mathcal{I}_{1+\lambda}$ .*

*Proof.* Suppose that  $\mathfrak{X}$  is based on an ordinal  $\Theta$ . Let  $\rho_\lambda$  be the rank-function with respect to  $\mathcal{T}_\lambda$ ; in view of Theorem 6.1, the rank with respect to  $\mathcal{I}_{1+\lambda}$  is  $\ell^{1+\lambda}$ .

We proceed to prove that  $\rho_\lambda = \ell^{1+\lambda}$  by induction on  $\lambda$ , with the base case  $\lambda = 0$  being immediate from the definitions and Lemma 5.1. For  $\lambda = \xi + 1$ , we use Lemma 8.1 to see that  $\mathfrak{X}_\lambda^+ = \dot{\mathfrak{X}}_\xi$ , and hence  $\mathcal{T}_\lambda$  is rank-preserving over  $\vec{\mathcal{T}}_\xi$ . By induction hypothesis,  $\mathcal{T}_\xi$  is a rank-preserving extension of  $\mathcal{I}_{1+\xi}$ .

Here we use a second induction on  $\ell^{1+\lambda}\vartheta$  to show that  $\rho_\lambda\vartheta = \ell^{1+\lambda}\vartheta$  for  $\vartheta < \Theta$ . Let  $U$  be a  $\vec{\mathcal{T}}_\lambda$ -neighborhood of  $\vartheta$  with  $\rho_\lambda(U) = [0, \rho_\lambda\vartheta)$ , so that  $U = V \cap (\alpha, \ell^{1+\xi}\vartheta]_{1+\xi}$  for some  $V \in \mathcal{T}_\xi$  and  $\alpha < \Theta$ . Let  $\delta < \ell^{1+\lambda}\vartheta$ . Then, there is  $\gamma \in (\alpha, \ell^{1+\xi}\vartheta]$  with  $\ell\gamma = \delta$ , since  $\ell$  maps intervals to initial segments.

But  $\gamma < \ell^{1+\xi}\vartheta$  and since by induction hypothesis  $\mathcal{T}_\xi$  is rank-preserving over  $\mathcal{I}_{1+\xi}$ , there is  $\eta \in V$  with  $\ell^{1+\xi}\eta = \rho_\xi\eta = \gamma$ . It follows that  $\eta \in U$  and, since  $U$  was arbitrary, that  $\rho_\lambda\vartheta > \delta$ . Since  $\delta$  was also arbitrary, we conclude that  $\rho_\lambda\vartheta \geq \ell^{1+\lambda}\vartheta$ . The inequality  $\rho_\lambda\vartheta \leq \ell^{1+\lambda}\vartheta$  follows from the fact that  $\mathcal{T}_\lambda$  refines  $\mathcal{I}_{1+\lambda}$ , and hence the two are equal.

If  $\lambda$  is a limit ordinal, first note that  $1 + \lambda = \lambda$ , which will simplify some expressions. Observe that  $\mathcal{T}_\lambda$  is rank-preserving over  $\mathcal{T}_\lambda^- = \bigsqcup_{\xi < \lambda} \mathcal{T}_\xi$ , so it suffices to compute ranks over  $\mathcal{T}_\lambda^-$ . Pick any basic  $\mathcal{T}_\lambda^-$ -neighborhood  $U$  of  $\vartheta$ , so that  $U \in \mathcal{T}_\xi$  for some  $\xi < \lambda$ , and  $\delta < \ell^\lambda\vartheta$ . We may assume  $U \subseteq [0, \vartheta]$ .



By induction on  $\xi < \lambda$ ,  $\rho_\xi U \supseteq [0, \ell^{1+\xi}\vartheta]$ , and hence there is  $\gamma \in U$  with  $\rho_\xi \gamma = e^{-(1+\xi)+\lambda}\delta < \vartheta$ , where the last inequality follows from Lemma 4.1. But then,  $\rho_\lambda \gamma \in \rho_\lambda U$ , and by induction on  $\gamma < \vartheta$  we have

$$\rho_\lambda \gamma = \ell^\lambda \gamma = \ell^{-(1+\xi)+\lambda} \ell^{1+\xi} \gamma = \ell^{-(1+\xi)+\lambda} e^{-(1+\xi)+\lambda} \delta = \delta.$$

Since  $U$  was arbitrary it follows that  $\rho_\lambda \vartheta \geq \ell^{1+\lambda} \vartheta$ , and hence the two are equal.  $\square$

Thus the analogue of Theorem 6.1 also holds for BG-spaces, although here we obtain  $\rho_\xi = \ell^{1+\xi}$ . In fact, the above result motivates our focusing on rank-preserving extensions of Icard spaces. Both BG-spaces and shifted Icard ambiances are examples of regular polytopologies, in the sense of the following definition:

**Definition 8.3** (regular space). *A  $\Lambda$ -space  $\mathfrak{X}$  with topologies  $\vec{\mathcal{T}}$  is regular if for all  $\lambda < \Lambda$ ,  $\mathcal{T}_\lambda$  is a limit-refinement of  $\mathcal{I}_{1+\lambda}$ .*

It remains to show that BG-spaces actually *exist*. This can be done via a simple but non-constructive proof:

**Lemma 8.4.** *Given ordinals  $\Theta, \Lambda$ , there exists a BG-space  $\langle \Theta, \langle \mathcal{T}_\lambda \rangle_{\lambda < \Lambda} \rangle$ .*

*Proof.* By a straightforward induction on  $\Lambda$  using Zorn's lemma to find a limit-maximal extension of  $\mathcal{T}_\Lambda^-$ .  $\square$

Below, a polytopology  $\langle X, \langle \mathcal{T}_\lambda \rangle_{\lambda < \Lambda} \rangle$  is *based* on a topological space  $\langle X, \mathcal{T} \rangle$  if  $\mathcal{T}_0$  is a limit-extension of  $\mathcal{T}$ .

**Lemma 8.5.** *Suppose that  $\mathfrak{X} = \langle X, \mathcal{T} \rangle$  is a scattered space,  $\mathfrak{Y} = \langle Y, \langle \mathcal{S}_\lambda \rangle_{\lambda < \Lambda} \rangle$  a BG-space and  $f : \mathfrak{X} \rightarrow \mathfrak{Y}_0$  a  $d$ -map.*

*Then, there exists a BG-space  $\mathfrak{X}'$  based on  $\mathfrak{X}$  such that  $f : \mathfrak{X}'_\lambda \rightarrow \mathfrak{Y}_\lambda$  is a  $d$ -map for all  $\lambda < \Lambda$ .*

*Proof.* We proceed by induction on  $\lambda$ , assuming we have constructed  $\mathcal{T}_\xi$  for  $\xi < \lambda$ . If  $\lambda = 0$ ,  $f : \mathfrak{X}_0 \rightarrow \mathfrak{Y}_0$  is a  $d$ -map by assumption so by Lemma 8.2 there is a limit-maximal extension  $\mathfrak{X}'$  of  $\mathfrak{X}$  such that  $f : \mathfrak{X}' \rightarrow \mathfrak{Y}_0$  is a  $d$ -map.

If  $\lambda = \xi + 1$  then by Lemma 8.1,  $\mathfrak{X}_\xi^+ = \dot{\mathfrak{X}}_\xi$  and by Lemma 5.6,  $f : \mathfrak{X}_\xi^+ \rightarrow \mathfrak{Y}_\xi^+$  is a  $d$ -map. Then, by Lemma 8.2, we can extend  $\mathfrak{X}_\xi^+$  to a limit-maximal space  $\mathfrak{X}_\lambda$  making  $f : \mathfrak{X}_\lambda \rightarrow \mathfrak{Y}_\lambda$  a  $d$ -map.

Finally, if  $\lambda \in \text{Lim}$ , we proceed as above, using Lemma 5.7.  $\square$

BG-spaces and Icard ambiances can sometimes be united into a single structure. We call these *idyllic ambiances*:

**Definition 8.4** (idyllic ambiance). *A shifted Icard ambiance  $\mathfrak{X}$  is idyllic if there is a BG polytopology on  $\mathfrak{X}$  such that, for all  $\lambda < \Lambda$ ,  $d_\lambda \upharpoonright \mathcal{A} = i_{1+\lambda} \upharpoonright \mathcal{A}$ .*

The purpose of these ambiances is to “kill two birds with one stone”, since any model based on an idyllic ambiance may be regarded both as an Icard model and a BG model. However, it will also be curious to observe that in our completeness proof, we shall construct BG-models and Icard models *with the same valuations*.

## 9 Reductive maps

The maps  $\ell^\Lambda$  are  $\ell^\lambda$ -invariant for  $\lambda < \Lambda$ , and we shall see that this is a very useful property. In fact, even a weaker, *local* version of this will turn out to be sufficient:

**Definition 9.1.** A map  $f : \Theta + 1 \rightarrow \Xi + 1$  is  $\Lambda$ -reductive if

1. it is a  $d_\Lambda$ -map and
2. for all  $\lambda < \Lambda$  and  $\xi \leq \Xi$  there is  $\delta < \xi$  such that, on  $f^{-1}(\delta, \xi]$ ,  $f$  is  $\ell^\lambda$ -invariant.

Reductive maps will be important in the study of idyllic ambiances. Note that on such ambiances,  $d_\lambda \upharpoonright \mathcal{A} = i_{1+\lambda} \upharpoonright \mathcal{A}$  must hold only for a *specific* BG-topology. But there are sets  $S$  such that  $d_\lambda S = i_{1+\lambda} S$  whenever  $d_\lambda$  is based on a BG-polytopology. We will say such a set  $S$  is  $\lambda$ -absolute.

**Lemma 9.1.** If  $f$  is  $\Lambda$ -reductive,  $\lambda < \Lambda$  and  $A$  is any set, then  $f^{-1}A$  is  $\lambda$ -absolute.

*Proof.* Let  $f : \Theta + 1 \rightarrow \Xi + 1$  be  $\Lambda$ -reductive and let  $\lambda < \Lambda$ .

Pick  $\delta < \Xi$  such that  $f \upharpoonright (\delta, \Xi]$  is  $\ell^\lambda$ -invariant. Then,

$$d_\lambda f^{-1}A = d_\lambda f^{-1}(A \cap [0, \delta]) \cup d_\lambda f^{-1}(A \cap (\delta, \Xi]).$$

By induction on  $\delta < \Xi$  we have that  $d_\lambda f^{-1}(A \cap [0, \delta]) = i_{1+\lambda} f^{-1}(A \cap [0, \delta])$ .

Meanwhile, letting  $B = \ell^\lambda(A \cap (\delta, \Xi])$ , by  $\ell^\lambda$ -invariance we see that  $f^{-1}(A \cap (\delta, \Xi]) = \ell^{-\lambda}B$ , so that by Lemma 5.5,

$$\begin{aligned} d_\lambda f^{-1}(A \cap (\delta, \Xi]) &= i_{1+\lambda} f^{-1}(A \cap (\delta, \Xi]) \\ &= \{\vartheta \leq \Theta : \ell^{1+\lambda}\vartheta > \min B\}, \end{aligned}$$

from which it follows that  $d_\lambda f^{-1}A = i_{1+\lambda} f^{-1}A$ . □

Another nice property of reductive maps is that they behave well with respect to extensions of limit topologies:

**Lemma 9.1.** Suppose that  $\Theta, \Xi$  are ordinals and  $\Lambda$  is a limit ordinal,  $f : \Theta_\Lambda \rightarrow \Xi_1$  is a  $\Lambda$ -reductive map and  $\langle \mathcal{T}_\lambda \rangle_{\lambda \leq \Lambda}$  is a regular polytopology on  $\Theta$  with  $\mathcal{T}_\Lambda = \bigsqcup_{\lambda < \Lambda} \mathcal{T}_\lambda$ . Let  $\mathfrak{X} = \langle \Theta, \vec{\mathcal{T}} \rangle$  be the resulting  $(\Lambda + 1)$ -space.

Then,  $f : \mathfrak{X}_\Lambda \rightarrow \Xi_1$  is a  $d$ -map.

*Proof.* Clearly  $f$  is continuous and pointwise discrete, so let us check that it is open.

Suppose that  $U$  is a  $\mathcal{T}_\Lambda$ -neighborhood of a point  $\vartheta < \Theta$ , so that it is a  $\mathcal{T}_\lambda$ -neighborhood of  $\vartheta$  for some  $\lambda < \Lambda$ . Pick an  $\mathcal{I}_\Lambda$ -neighborhood  $D$  of  $\vartheta$  such that  $f$  is  $\ell^{1+\lambda}$ -invariant on  $D$  and  $\ell^{1+\lambda}(D \setminus \{\vartheta\}) = [0, \ell^{1+\lambda}\vartheta]$ ; the first condition can be met because  $f$  is  $\Lambda$ -reductive, the second by Lemmas 5.2 and 6.1.

We claim that  $f(U \cap D) = f(D)$ . Indeed, if  $\zeta \in f(D)$ , then  $\zeta = f(\delta)$  for some  $\delta \in D$ . But since  $\rho_\lambda \vartheta = \ell^{1+\lambda}\vartheta \geq \rho_\lambda \delta$ , there is some  $\delta' \in U \cap D$  with

$$\ell^{1+\lambda}\delta' = \rho_\lambda \delta' = \rho_\lambda \delta = \ell^{1+\lambda}\delta,$$

and hence  $f(\delta') = f(\delta) = \zeta$ . Since  $\zeta$  was arbitrary, the claim follows.

Now,  $f$  is a  $d$ -map with respect to  $\mathcal{I}_\Lambda$ , so that  $f(D)$  (and hence  $f(U \cap D)$ ) is open, as desired.  $\square$

Not all reductive maps are given by hyperlogarithms. Let us now construct another interesting example. Here we will work with *fundamental sequences*; that is, we assume that to each countable limit ordinal  $\xi$  we have assigned a sequence of ordinals  $\langle \xi[n] \rangle_{n < \omega}$  with the property that  $\xi[n] < \xi[n+1]$  for all  $n$  and  $\xi = \lim_{\xi \rightarrow \omega} \xi[n]$ . If  $\xi = \zeta + 1$ , we will write  $\zeta = \xi[n]$  for all  $n$ .

Suppose that  $\Lambda$  is infinite and additively indecomposable. If  $\vartheta < e^\Lambda \Theta$  is any ordinal that is not in the range of  $e^\Lambda$ , there exists a value of  $N$  such that  $\ell^{\Lambda[N]}\vartheta \leq e^\Lambda(\Theta[N])$ ; if  $\Theta$  is a successor ordinal this is essentially Lemma 4.3, otherwise  $e^\Lambda \Theta = \lim_{n \rightarrow \omega} e^\Lambda(\Theta[n])$  (since  $e^\Lambda$  is normal), and hence for some value of  $N$  we already have that  $\vartheta < e^\Lambda(\Theta[n])$ . We will denote the smallest such value of  $N$  by  $N_\Lambda^\Theta(\vartheta)$ .

**Lemma 9.2.** *If  $\Theta, \Lambda$  are ordinals such that  $\Lambda$  is infinite and additively indecomposable then for every  $N > 1$ , the set*

$$\Delta_\Lambda^\Theta[N] = \{\vartheta < \Theta : N_\Lambda^\Theta(\vartheta) = N\}$$

*is  $(\Lambda[N] + 1)$ -simple.*

*Further,*

$$\ell^{\Lambda[N]}\Delta_\Lambda^\Theta[N] = [0, e^\Lambda(\Theta[N])]. \quad (1)$$

*Proof.* Let  $\alpha = e^\Lambda(\Theta[N])$  and  $\beta = e^\Lambda(\Theta)$ .

We have that

$$\Delta_\Lambda^\Theta[N] = \bigcap_{n < N} (\alpha, \beta)_{\Lambda[n]} \cap [0, \alpha]_{\Lambda[N]},$$

which is  $(\Lambda[N] + 1)$ -simple.

Now, let  $\delta \leq \alpha$  and consider the simple function given by  $r(\lambda[n]) = \alpha + 1$  for  $n < N$  and  $r(\lambda[N+1]) = \delta$ .

Then, by Lemma 4.4,  $\lceil r \rceil \in \Delta_\Lambda^\Theta[N]$  and  $\ell^{\Lambda[N]}\lceil r \rceil = \delta$ . Since  $\delta \leq \alpha = e^\Lambda(\Theta[N])$  was arbitrary, we conclude that (1) holds.  $\square$

Below, a family of sets  $\mathcal{U}$  forms a *neighborhood base* for  $x$  if every element of  $\mathcal{U}$  is a neighborhood of  $x$  and for every neighborhood  $V$  of  $x$  there exists  $U \in \mathcal{U}$  such that  $V \subseteq U$ .

**Lemma 9.3.** *Let  $\Lambda$  be countable and additively indecomposable and  $\Theta$  be any ordinal.*

*Then, the sets*

$$D_\Lambda^\Theta[N] = (e^\Lambda(\Theta[N]), e^\Lambda(\Theta)]_{\Lambda[N]}$$

*form a  $\Lambda$ -neighborhood base for  $e^\Lambda\Theta$ .*

*Further,  $D_\Lambda^\Theta[N] = \{e^\Lambda\Theta\} \cup \bigcup_{n>N} \Delta_\Lambda^\Theta[n]$ .*

*Proof.* Clearly  $e^\Lambda\Theta \in D_\Lambda^\Theta[N]$  for all  $N$  and  $D_\Lambda^\Theta[N]$  is open.

Now, let  $B_r(e^\Lambda\Theta)$  be any basic  $\Lambda$ -neighborhood of  $e^\Lambda\Theta$ ; we need to find a smaller neighborhood of the form  $D_\Lambda^\Theta[N]$ . Let us consider two cases.

First assume  $\Theta = \Theta' + 1$ , so that  $\Theta[n] = \Theta'$  for all  $n$ .

We have by Lemma 4.1.3 that

$$e^\Lambda\Theta = \lim_{n \rightarrow \omega} e^{\Lambda[n]}(e^\Lambda\Theta' + 1)$$

and hence for some value of  $N$  we have that  $e^{\Lambda[N]}(e^\Lambda\Theta' + 1) > r(\xi)$  for all  $\xi$ .

Let  $M > N$  be large enough so that  $\Lambda[M] > \xi + \Lambda[N]$  for all  $\xi \in \text{supp}(r)$ . Then, we claim that  $D_\Lambda^\Theta[M] \subseteq B_r(\vartheta)$ ; for indeed, if  $\xi \in \text{supp}(r)$  and  $\delta \in D_\Lambda^\Theta[M]$  then

$$\ell^{\Lambda[M]}\delta = \ell^{-\xi+\Lambda[M]}\ell^\xi\delta \geq e^\Lambda(\Theta') + 1$$

so that by Lemma 4.1

$$\ell^\xi\delta \geq e^{-\xi+\Lambda[M]}(e^\Lambda(\Theta') + 1) \geq e^{\Lambda[N]}(e^\Lambda(\Theta') + 1) \geq r(\xi).$$

Thus,  $D_\Lambda^\Theta[M] \subseteq B_r(e^\Lambda\Theta)$ .

If  $\Theta$  is a limit ordinal, the argument is somewhat simpler; pick  $M$  so that  $\Theta[M] > r(\xi)$  for all  $\xi$ . Then, it is easy to check that  $D_\Lambda^\Theta[M] \subseteq B_r(e^\Lambda\Theta)$ .

Finally, to see that  $D_\Lambda^\Theta[N] = \{e^\Lambda\Theta\} \cup \bigcup_{n>N} \Delta_\Lambda^\Theta[n]$ , note that every  $\xi < e^\Lambda\Theta$  lies in  $\Delta_\Lambda^\Theta[n]$  for some  $n$ , and  $n > N$  if and only if  $\xi \in D_\Lambda^\Theta[N]$ .  $\square$

On  $[0, e\Theta]$  we shall consider a different class of neighborhoods. Define

$$\sigma_\Theta[N] = \sum_{n<N} (e(\Theta[n]) + 1)$$

and  $\Sigma_\Theta[N] = [\sigma_\Theta[N], \sigma_\Theta[N+1])$ .

Similarly, define  $S_\Theta[N] = [\sigma_\Theta[N], e\Theta]$ .

Then we have that:

**Lemma 9.4.** *The sets  $\{\Sigma_\Theta[n] : n < \omega\}$  form a partition of  $(0, e\Theta)$  into 1-open sets.*

*Further, the sets  $\{S_\Theta[n] : n < \omega\}$  form a 1-neighborhood base for  $e\Theta$ , and*

$$S_\Theta[N] = \{e(\Theta)\} \cup \bigcup_{n>N} \Sigma_\Theta[n].$$

*Proof.* Note that despite its formal appearance the set  $\Sigma_\Theta[N]$  is always open since  $\sigma_\Theta[N]$  is always a successor ordinal. The rest of the claims are obvious if we show that  $\langle \sigma_\Theta[n] \rangle_{n < \omega}$  is unbounded in  $e\Theta$ .

Here we consider two cases; if  $\Theta = \Theta' + 1$ , then  $\sigma_\Theta[n] = e(\Theta')n$  for all  $n < \omega$ , and

$$e(\Theta' + 1) = \omega^{\Theta'+1} = \lim_{n \rightarrow \omega} \omega^{\Theta'} n + 1 = \lim_{n \rightarrow \omega} (e(\Theta') + 1)n.$$

But  $(e(\Theta') + 1)N = \sigma_\Theta[N]$ .

Meanwhile, if  $\Theta \in \text{Lim}$ , then  $\sigma_\Theta[N + 1] = e(\Theta[N]) + 1$  (all previous terms cancel) and  $e\Theta = \lim_{n \rightarrow \omega} e(\Theta[N]) + 1$ .  $\square$

With this we may define the following maps:

**Definition 9.2.** *Given countable ordinals  $\Lambda, \Theta$  such that  $\Lambda$  is infinite and additively indecomposable, we will define a function*

$$\mathfrak{d}_\Lambda^\Theta : (e^{1+\Lambda}(\Theta) + 1) \rightarrow (e(\Theta) + 1)$$

assuming  $\mathfrak{d}_\Lambda^{\Theta'}$  is defined whenever  $\Theta' < \Theta$ .

First define  $\mathfrak{d}_\Lambda^\Theta e^{1+\Lambda}\Theta = e\Theta$ .

Then, for  $\xi < e^{1+\Lambda}\Theta$ , set  $N = N_\Lambda^\Theta(\xi)$  and

$$\mathfrak{d}_\Lambda^\Theta \xi = \sigma_\Theta[N] + \mathfrak{d}_\Lambda^{\Theta[N]} \ell^{\lambda[N]} \xi.$$

The maps  $\mathfrak{d}_\Lambda^\Theta$  we have defined are  $\Lambda$ -reductive, but proving this will require several steps. Let us begin with a useful technical lemma.

**Lemma 9.5.** *Given an ordinal  $\Theta$  and a limit ordinal  $\Lambda$ ,*

1.  $\mathfrak{d}_\Lambda^\Theta[0, e^\Lambda\Theta] = [0, e\Theta]$
2. if  $N > 0$ ,  $\mathfrak{d}_\Lambda^\Theta \Delta_\Lambda^\Theta[N] = \Sigma_\Theta[N]$ .

*Proof.* Assume both claims are true for  $\Theta'$  when  $\Theta' < \Theta$ .

Note that the first claim is trivial when  $\Theta = 0$  because then both sides of the equality are the singleton  $\{0\}$ , so we may assume  $\Theta > 0$ .

We shall begin by checking the inclusion

$$\mathfrak{d}_\Lambda^\Theta \Delta_\Lambda^\Theta[N] \subseteq \Sigma_\Theta[N].$$

For  $\vartheta \in \Delta_\Lambda^\Theta[N]$  we see that

$$\mathfrak{d}_\Lambda^\Theta \vartheta = \sigma_\Theta[N] + \mathfrak{d}_\Lambda^{\Theta[N]} \ell^{\lambda[N]} \vartheta \in [\sigma_\Theta[N], \sigma_\Theta[N + 1]],$$

where the last step uses Claim 1 by induction on  $\Theta[N] < \Theta$ , given that  $\ell^{\lambda[N]} \vartheta \in [0, e^\Lambda(\Theta[N])]$ .

From this it easily follows that  $\mathfrak{d}_\Lambda^\Theta[0, e^\Lambda\Theta] \subseteq [0, e\Theta]$ , since  $\mathfrak{d}_\Lambda^\Theta e^\Lambda\Theta = e\Theta \in [0, e\Theta]$ , while for  $\vartheta < e^\Lambda\Theta$ , we set  $N = N_\Lambda^\Theta(\vartheta)$  and by Claim 2 see that

$$\mathfrak{d}_\Lambda^\Theta \vartheta \in \Sigma_\Theta[N] \subseteq [0, e\Theta].$$

Now let us check that

$$\Sigma_\Theta[N] \subseteq \mathfrak{d}_\Lambda^\Theta \Delta_\Lambda^\Theta[N].$$

Let  $\xi = \sigma_\Theta[N] + \xi'$  with  $\xi' \leq e(\Theta[N])$ . Then, using Claim 1 by induction on  $\Theta[N] < \Theta$ ,  $\xi' = \mathfrak{d}_\Lambda^{\Theta[N]} \gamma'$  for some  $\gamma' \leq e^\Lambda(\Theta[N])$ ; meanwhile, by Lemma 9.2,  $\ell^{\Lambda[N]} \Delta_\Lambda^\Theta[N] = [0, e^\Lambda(\Theta[N])]$ , hence  $\gamma' = \ell^{\Lambda[N]} \gamma$  for some  $\gamma \in \Delta_\Lambda^\Theta[N]$ . It immediately follows that  $\xi = \mathfrak{d}_\Lambda^\Theta \gamma$ .

To check the remaining inclusion of Claim 1, consider  $\xi \in [0, e\Theta]$ . If  $\xi = e\Theta$ , then evidently  $\xi = \mathfrak{d}_\Lambda^\Theta e^\Lambda(\Theta)$ . Otherwise,  $\xi \in [\sigma_\Theta[N], \sigma_\Theta[N+1])$  for some  $N$ . But by Claim 2,

$$\xi \in \mathfrak{d}_\Lambda^\Theta \Delta_\Lambda^\Theta[N] \subseteq \mathfrak{d}_\Lambda^\Theta [0, e^\Lambda \Theta],$$

as required.  $\square$

**Lemma 9.6.** *Given countable ordinals  $\Theta, \Lambda$  such that  $\Lambda$  is infinite and additively indecomposable,*

$$\mathfrak{d}_\Lambda^\Theta : (e^\Lambda(\Theta) + 1)_\Lambda \rightarrow (e(\Theta) + 1)_1$$

*is an onto  $d$ -map.*

*Proof.* Assume the claim is true for all  $\Theta' < \Theta$ . Note that surjectivity is already proven in Lemma 9.5. Also, the case  $\Theta = 0$  is trivial because then both sides of the equality are the singleton  $\{0\}$ , so that we may assume  $\Theta > 0$ .

Pick  $\vartheta \leq e^\Lambda(\Theta)$ . If  $\vartheta < e^\Lambda(\Theta)$ ,  $\vartheta \in \Delta_\Lambda^\Theta[N]$  for some  $N$ , and since these sets are all open by Lemma 9.2, it suffices to observe that  $\mathfrak{d}_\Lambda^\Theta \restriction \Delta_\Lambda^\Theta[N]$  is a  $d$ -map. But it is equal to  $\mathfrak{d}_\Lambda^{\Theta[N]} \ell^{\Lambda[N]}$ , which by induction on  $\Theta[N] < \Theta$  and Lemma 6.2 is a composition of  $d$ -maps. Hence it is open and continuous near  $\vartheta$ .

Otherwise,  $\vartheta = e^\Lambda(\Theta)$ . Here we claim that  $\mathfrak{d}_\Lambda^\Theta D_\Lambda^\Theta[N] = S_\Theta[N]$ , from which openness and continuity are immediate.

We have that  $D_\Lambda^\Theta[N] = \{e^\Lambda \Theta\} \cup \bigcup_{n>N} \Delta_\Lambda^\Theta[n]$ , whereas  $S_\Theta[N] = \{e\Theta\} \cup \bigcup_{n>N} \sigma_\Theta[n]$ , and thus

$$\begin{aligned} \mathfrak{d}_\Lambda^\Theta D_\Lambda^\Theta[N] &= \mathfrak{d}_\Lambda^\Theta \{e^\Lambda \Theta\} \cup \bigcup_{n>N} \mathfrak{d}_\Lambda^\Theta \Delta_\Lambda^\Theta[n] \\ &= \{e\Theta\} \cup \bigcup_{n>N} \Sigma_\Theta[n] \\ &= S_\Theta[N], \end{aligned}$$

where the second equality follows from Lemma 9.5.

To check that  $\mathfrak{d}_\Lambda^\Theta$  is pointwise discrete, pick  $\xi \leq e\Theta$ . If  $\xi = e\Theta$  then  $(\mathfrak{d}_\Lambda^\Theta)^{-1} \xi = \{e^\Lambda \Theta\}$ , which is discrete.

Otherwise,  $\xi \in \Sigma_\Theta[N]$  for some  $N$  and thus  $(\mathfrak{d}_\Lambda^\Theta)^{-1} \xi = \Delta_\Lambda^\Theta[N] \cap (\mathfrak{d}_\Lambda^\Theta)^{-1} \xi$ , which is discrete as  $\mathfrak{d}_\Lambda^\Theta \restriction \Delta_\Lambda^\Theta[N]$  is a  $d$ -map.  $\square$

**Lemma 9.7.** *For any countable ordinal  $\Lambda$ , the map  $\mathfrak{d}_\Lambda^\Theta$  is  $\Lambda$ -reductive.*

*Proof.* By Lemma 9.6, it suffices to prove that  $\mathfrak{d}_\Lambda^\Theta$  is locally  $\ell^\Lambda$ -invariant for all  $\lambda < \Lambda$ . For all  $\vartheta < e^{1+\Lambda}\Theta$  this follows by induction hypothesis using the fact that  $\vartheta \in \Delta_\Lambda^\Theta[N]$  for some  $N$ , otherwise we use the fact that on  $D_\Lambda^\Theta[N]$ ,  $\mathfrak{d}_\Lambda^\Theta$  is  $\Lambda[N]$ -invariant.  $\square$

Let us conclude this section by extending  $\mathfrak{d}_\Lambda^\Theta$  to the case where  $\Lambda$  is not necessarily additively indecomposable.

**Theorem 9.1.** *Given countable ordinals  $\Theta, \Lambda$  with  $\Lambda \geq 0$  there exists a  $\Lambda$ -reductive surjection*

$$\mathfrak{d}_\Lambda^\Theta : (e^{1+\Lambda}(\Theta) + 1)_{1+\Lambda} \rightarrow (e(\Theta) + 1)_1.$$

*Proof.* If  $\Lambda$  is infinite and additively indecomposable we have already defined  $\mathfrak{d}_\Lambda^\Theta$ .

Otherwise, set  $\mathfrak{d}_0^\Theta = \text{id}$ ,  $\mathfrak{d}_1^\Theta = \ell$ , and if  $\Lambda = \omega^\alpha + \beta$  with  $\beta < \Lambda$  define  $\mathfrak{d}_\Lambda^\Theta = \mathfrak{d}_{\omega^\alpha}^{\beta\Theta} \mathfrak{d}_\beta^\Theta$ , which is easily seen to be  $\Lambda$ -reductive.  $\square$

## 10 Operations on ambiances

In this section we shall review some operations, many of which were introduced in [5], that may be used to construct new provability ambiances from existing ones. First, let us observe that ambiances may be “pulled back”.

If  $\mathfrak{X}$  is an  $(\alpha + \beta)$ -BG space and  $\mathfrak{Y}$  is a  $\beta$ -BG space,  $f : \mathfrak{X} \rightarrow \mathfrak{Y}$  is an  $\alpha$ -lift if it is  $\alpha$ -reductive and, for all  $\delta < \beta$ ,  $f : \mathfrak{X}_{\alpha+\delta} \rightarrow \mathfrak{Y}_\delta$  is a  $d$ -map both with respect to the BG topologies and the shifted Icard topologies.

**Lemma 10.1.** *Suppose that  $\mathfrak{X}$  is a  $(\xi + \zeta)$ -BG-space and  $\mathfrak{Y}$  is an idyllic  $\zeta$ -ambiance with algebra  $\mathcal{A}$ .*

*Suppose further that  $f : \mathfrak{X}_{\xi+\delta} \rightarrow \mathfrak{Y}_\delta$  is a  $\xi$ -lift.*

*Then,  $f^{-1}\mathcal{A}$  is an idyllic algebra on  $\mathfrak{X}$ .*

*Proof.* To see that  $d_{\xi+\delta} \upharpoonright f^{-1}\mathcal{A} = i_{1+\xi+\delta} \upharpoonright f^{-1}\mathcal{A}$ , note that for  $A \in \mathcal{A}$ ,

$$d_{\xi+\delta} f^{-1}A = f^{-1}d_{\xi+\delta}A = f^{-1}i_{1+\xi+\delta}A = i_{1+\xi+\delta}f^{-1}A.$$

That  $d_\delta \upharpoonright f^{-1}\mathcal{A} = i_{1+\delta} \upharpoonright f^{-1}\mathcal{A}$  for  $\delta < \alpha$  follows from Lemma 9.1 since  $f^{-1}A$  is always  $\delta$ -absolute.  $\square$

One of the basic ways of including topological spaces into a larger space is by their *topological sum*. In the case of ordinal spaces, this is closely tied to the sum of ordinals, as has already been observed in [5]:

**Definition 10.1.** *Given families of sets  $\mathcal{A} \subseteq \mathcal{P}(\eta)$  and  $\mathcal{B} \subseteq \mathcal{P}(\vartheta)$ , where  $\eta, \vartheta$  are ordinals, we define  $\mathcal{A} \oplus \mathcal{B}$  to be the family of subsets of  $\eta + \vartheta$  of the form*

$$S = S_0 \cup (\eta + S_1),$$

*with  $S_0 \in \mathcal{A}$  and  $S_1 \in \mathcal{B}$ .*

The following lemma is standard and easy to check:

**Lemma 10.2.** *If  $\eta, \vartheta$  are ordinals and  $\mathcal{A} \subseteq \mathcal{P}(\eta), \mathcal{B} \subseteq \mathcal{P}(\vartheta)$  are topologies, then  $\mathcal{A} \oplus \mathcal{B}$  is a topology on  $\eta + \vartheta$ .*

*If  $\eta$  is a successor and both topologies are Icard or BG, then  $\mathcal{A} \oplus \mathcal{B}$  is Icard or BG, respectively.*

In view of this we define, given  $\Lambda$ -polytopologies  $\mathfrak{X} = \langle \eta, \vec{\mathcal{T}} \rangle$  and  $\mathfrak{Y} = \langle \vartheta, \vec{\mathcal{S}} \rangle$ , the sum

$$\mathfrak{X} \oplus \mathfrak{Y} = \langle \eta + \vartheta, \langle \mathcal{T}_\lambda \oplus \mathcal{S}_\lambda \rangle_{\lambda < \Lambda} \rangle.$$

We may also apply the sum operation to  $d$ -algebras:

**Lemma 10.3.** *Suppose  $\mathfrak{X} = \langle \eta + 1, \vec{\mathcal{T}}, \mathcal{A} \rangle$  and  $\mathfrak{Y} = \langle \vartheta, \vec{\mathcal{S}}, \mathcal{A} \rangle$  are ambiances.*

*Then,  $\mathfrak{X} \oplus \mathfrak{Y}$  equipped with  $\mathcal{A} \oplus \mathcal{B}$  is also an ambiance.*

*Further, if both ambiances are idyllic, then so is the corresponding sum.*

We shall not present a proof, as this result is fairly obvious once we observe that the all relevant operations may be carried out independently within the two disconnected subspaces.

The last major topological construction needed for the completeness proof is the notion of a  $d$ -product. Let  $\mathfrak{X} = \langle [0, \eta], \langle \mathcal{T}_\lambda \rangle_{\lambda < \Lambda} \rangle$  and  $\mathfrak{Y} = \langle [0, \vartheta], \langle \mathcal{S}_\lambda \rangle_{\lambda < \Lambda} \rangle$  be polytopologies and define  $\eta \otimes_d \vartheta = -1 + (1 + \eta)\omega(1 + \vartheta)$ .

Let  $G_1$  be the set of those  $\xi \leq \eta \otimes_d \vartheta$  of the form  $(1 + \eta)\omega(1 + \xi_1)$  and  $G_0$  be its complement. Every  $\xi \in G_0$  can be written uniquely in the form

$$\xi = -1 + (1 + \eta)\alpha + (1 + \xi_0)$$

with  $\xi_0 \leq \eta$ , and we define  $\pi_0 : G_0 \rightarrow [0, \eta]$  by setting  $\pi_0(\xi) = \xi_0$ . Similarly, define  $\pi_1 : G_1 \rightarrow [0, \vartheta]$  by  $\pi_1((\eta + 1)\omega(1 + \xi_1)) = \xi_1$ ; we should remark that  $\pi_1$  is in fact a bijection.

With this, we may define the  $d$ -product of spaces:

**Definition 10.2** ( $d$ -product). *Let  $\mathfrak{X} = \langle [0, \eta], \langle \mathcal{T}_\lambda \rangle_{\lambda < \Lambda} \rangle$  and  $\mathfrak{Y} = \langle [0, \vartheta], \langle \mathcal{S}_\lambda \rangle_{\lambda < \Lambda} \rangle$  be polytopologies.*

*For  $\lambda < \Lambda$ , define a topology  $\mathcal{O}_\lambda$  on  $[0, \eta \otimes_d \vartheta]$  by letting  $U$  be open if, for every  $\xi \in U$ , either*

1.  $\xi \in G_0$  and there are  $\delta < \xi$  and  $V \in \mathcal{T}_\lambda$  with  $\xi \in (\delta, \xi] \cap \pi_0^{-1}V \subseteq U$ ,
2.  $\xi \in G_1$ ,  $\pi_1\xi \in \text{Lim}$  and there is  $V \in \mathcal{S}_\lambda$  such that  $\xi \in \pi_1^{-1}V \subseteq U$  or
3.  $\xi \in G_1$ ,  $\pi_1\xi$  is a successor and either
  - (a)  $\lambda > 0$  or
  - (b)  $\lambda = 0$  and there is  $\zeta < \xi$  such that  $(\zeta, \xi] \subseteq U$ .

*We denote the resulting space  $\langle [0, \eta \otimes_d \vartheta], \langle \mathcal{O}_\lambda \rangle_{\lambda < \Lambda} \rangle$  by  $\mathfrak{X} \otimes_d \mathfrak{Y}$ .*

This definition will be sufficient for our purposes, but the  $d$ -product of spaces is treated with much more generality and detail in [5], which gives them a slightly different presentation. There, the following properties are established:



**Lemma 10.1.** *If  $\mathfrak{X} = \langle [0, \eta], \langle \mathcal{T}_\lambda \rangle_{\lambda < \Lambda} \rangle$  and  $\mathfrak{Y} = \langle [0, \vartheta], \langle \mathcal{S}_\lambda \rangle_{\lambda < \Lambda} \rangle$  are BG-spaces and  $\mathfrak{Z} = \mathfrak{X} \otimes_d \mathfrak{Y}$ , then*

1.  $\mathfrak{Z}$  is a BG-space,
2. both projections  $\pi_i$  are  $d$ -maps with respect to all  $\lambda < \Lambda$  and
3. for every  $\xi \in [1, \eta]$ ,  $\pi_0^{-1}\xi$  is 0-dense in  $G_1$ .

Note, however, that the  $d$ -product of Icard spaces is not an Icard space. Thus the analogue of Lemma 10.1 does not hold. Instead, we have the following:

**Lemma 10.2.** *Suppose that  $\eta, \vartheta$  are ordinals,  $\Theta = \eta \otimes_d \vartheta$  and  $G_0, G_1$  are the components of  $\Theta$  with projections  $\pi_i$ .*

*Then,*

1.  $G_0$  is 0-open in  $\widehat{\mathfrak{C}}_\Lambda^{\Theta+1}$  and, for all  $\lambda < \Lambda$ ,

$$\pi_0 : \langle G_0, \mathcal{I}_{1+\lambda} \rangle \rightarrow \langle \eta + 1, \mathcal{I}_{1+\lambda} \rangle$$

*is a  $d$ -map<sup>5</sup>.*

2. Given  $U \subseteq [0, \vartheta]$ ,  $U$  is open if and only if  $G_0 \cup \pi_1^{-1}U$  is 0-open.
3.  $G_1$  is 1-open and for all  $\lambda$ ,

$$\pi_1 : \langle G_1, \mathcal{I}_{1+\lambda} \rangle \rightarrow \langle \vartheta + 1, \mathcal{I}_{1+\lambda} \rangle$$

*is a  $d$ -map.*

*Proof.* Recall from Section 6 that we may write intervals of the form  $[0, \beta]_\lambda$  as  $(-1, \beta]_\lambda$ ; we shall use this convention throughout the proof in order to treat all cases uniformly.

**1.** Observe that  $G_0$  is a union of intervals of the form  $J_\alpha = (\alpha, \alpha + 1 + \eta)$ , where  $\alpha = -1 + (1 + \eta)\alpha'$ ; this shows that  $G_0$  is open. Further,  $\pi_0 \upharpoonright J_\alpha$  is simply  $\xi \mapsto -\alpha + \xi$ , which is obviously a  $d$ -map with respect to all Icard topologies.

**2.** First assume that  $G_0 \cap \pi_1^{-1}U$  is open, and let  $\xi \in U$ . By assumption, there is a neighborhood  $(\zeta, \pi_1^{-1}\xi] \subseteq G_0 \cap \pi_1^{-1}U$  of  $\pi_1^{-1}\xi$ . Let  $\delta \in U$  be the least ordinal such that  $\pi_1^{-1}\delta \in (\zeta, \pi_1^{-1}\xi]$ . We claim that  $\delta$  is a successor or zero; for otherwise, given that  $\pi_1^{-1} \equiv \alpha \mapsto (1 + \eta)\omega(1 + \alpha)$  is a normal function we would have  $\pi_1^{-1}\delta' > \zeta$  for  $\delta' < \delta$  large enough, contradicting the minimality of  $\delta$ .

Thus  $\delta = \delta' + 1$ , and it is easy to see using the monotonicity of  $\pi_1$  that  $(\delta', \xi] \subseteq U$ . Since  $\xi$  was arbitrary, we conclude that  $U$  is open.

Meanwhile, if  $U$  is open, let  $\xi \in G_0 \cup \pi_1^{-1}U$ . If  $\xi \in G_0$ , then we know that  $G_0$  is already open by Item 1, and hence there is nothing to prove. So we assume that  $\xi \in \pi_1^{-1}U$ , and since  $U$  is open there is  $\zeta < \pi_1\xi$  with  $(\zeta, \pi_1\xi] \subseteq U$ .

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<sup>5</sup> $\langle G_0, \mathcal{I}_\lambda \rangle$  should be understood as  $G_0$  under the subspace topology induced by  $\mathcal{I}_\lambda$ .

We then claim that  $(\pi_1^{-1}\zeta, \xi] \subseteq G_0 \cup \pi_1^{-1}U$ . To see this, pick  $\delta \in (\pi_1^{-1}\zeta, \xi]$ . If  $\delta \in G_0$  there is nothing to prove, otherwise since  $\pi_1$  is increasing and  $\pi_1^{-1}\zeta < \delta < \xi$  we have that  $\zeta < \pi_1\delta < \pi_1\xi$  and thus  $\pi_1\delta \in (\zeta, \pi_1\xi] \subseteq U$ . Since  $\delta$  was arbitrary we conclude that  $(\pi_1^{-1}\zeta, \xi] \subseteq G_0 \cup \pi_1^{-1}U$  and the latter is open, as required.

**3.** Any ordinal  $\xi \in G_1$  is of the form  $(1 + \eta)\omega(1 + \zeta)$ , with  $\zeta \leq \vartheta$ . Write  $1 + \eta = \omega^\alpha + \beta$  with  $\beta < \eta$  and  $1 + \zeta = \gamma + \omega^\delta$ . Then, we see that

$$\xi = (\omega^\alpha + \beta)\omega(\gamma + \omega^\delta) = \omega^{\alpha+1}\gamma + \omega^{\alpha+1+\delta}. \quad (2)$$

Note further that  $\rho[0, \eta] = \alpha + 1$ . Thus for every element  $\xi$  of  $G_0$ ,  $\ell\xi \leq \alpha$ , while for every  $\xi \in G_1$  we have  $\ell\xi > \alpha$ , from which it follows that  $G_1 = (\alpha, \rho[0, \Theta])_1$  is 1-open.

Let us now show that  $\pi_1$  is open. For this, we claim that, for all  $\lambda < \Lambda$  and all  $\zeta < \xi < \vartheta$ , there are  $\zeta', \xi'$  with  $\pi_1(\zeta, \xi]_\lambda = (\zeta', \xi']_\lambda$ .

For  $\lambda = 0$ , we use the fact that  $\pi_1^{-1}$  is a normal function to see that  $\pi_1(\zeta, \xi]$  must be an interval of the form  $(\zeta', \xi']$ .

For  $\lambda = 1$ , observing (2) we see that  $\ell\pi_1\xi = -(\alpha + 1) + \ell\xi$ . Thus  $G_1 \cap (\zeta, \xi]_1$  is either empty or equal to  $G_1 \cap (\zeta'', \xi]_1$  for some  $\zeta'' \geq \alpha$ , and

$$\pi_1(G_1 \cap (\zeta'', \xi]_1) = [0, \vartheta] \cap (-(\alpha + 1) + \zeta'', -(\alpha + 1) + \xi]_1;$$

here,  $\zeta'$  is  $-1$  whenever  $\zeta'' = \alpha$ . Note that the right-to-left inclusion uses the fact that  $\pi_1$  is surjective.

Meanwhile, for  $\lambda > 1$ , once again we use (2) to see that  $\ell^\lambda\pi_1\xi = \ell^\lambda\xi$  and thus  $\pi_1(G_1 \cap (\zeta, \xi]_\lambda) = [0, \vartheta] \cap (\zeta, \xi]_\lambda$ .

From this and the fact that  $\pi_1$  is a bijection it follows that it is  $\lambda$ -open for any  $\lambda < \Lambda$ , since the image of any basic open is open:

$$\pi_1\left(G_1 \cap \bigcap_{i < I} (\zeta_i, \xi_i]_{\sigma_i}\right) = [0, \vartheta] \cap \bigcap_{i < I} \pi_1(\zeta_i, \xi_i]_{\sigma_i} = [0, \vartheta] \cap \bigcap_{i < I} (\zeta'_i, \xi'_i]_{\sigma_i}.$$

The continuity of  $\pi_1$  is proven analogously, and the fact that it is pointwise discrete follows from the fact that it is injective.  $\square$

We should remark that by Lemma 10.2.1,  $G_0$  is 0-open in  $\mathfrak{X} \otimes_d \mathfrak{Y}$  given that the  $d$ -product topologies are regular, and similarly by Lemma 10.2.3,  $G_1$  is 1-open. Meanwhile, by Lemma 10.1.3,  $\pi_0^{-1}\xi$  is always dense in  $G_1$  with respect to  $\mathcal{I}_1$ .

With these ingredients, we define the  $d$ -product of algebras:

**Definition 10.3** ( $d$ -product of algebras). *Given  $d$ -algebras  $\mathcal{A}, \mathcal{B}$  based on  $\eta + 1$ ,  $\vartheta + 1$ , we define  $\mathcal{A} \otimes_d \mathcal{B}$  to be the algebra of all sets  $S$  of the form*

$$S = \pi_0^{-1}(S_0) \cup \pi_1^{-1}(S_1),$$

where  $S_0 \in \mathcal{A}$  and  $S_1 \in \mathcal{B}$ .

Of course, we would like for the  $d$ -product of algebras to be itself a  $d$ -algebra. The next lemma will be useful in showing this.

**Lemma 10.3.** *Let  $\mathfrak{X}, \mathfrak{Y}$  be polytopologies based on ordinals  $\eta + 1, \vartheta + 1$ , respectively. Let  $d_\xi$  denote the  $\xi$ -derived set operator on  $\mathfrak{Y}$  and  $d'_\xi$  on  $\mathfrak{X} \otimes_d \mathfrak{Y}$ .*

*Then, for any  $E \subseteq \vartheta$  and  $\lambda < \Lambda$ ,  $d'_\lambda \pi_1^{-1}E = \pi_1^{-1}d_\lambda E$  and  $i_{1+\lambda} \pi_1^{-1}E = \pi_1^{-1}i_{1+\lambda}E$ .*

*Proof.* Assume first that  $\xi \in d'_\lambda \pi_1^{-1}(E)$ . Note that this immediately implies that  $\xi \notin G_0$ , since the latter is 0-open.

Then, for every  $\lambda$ -neighborhood  $U$  of  $\xi$  there is  $\zeta \neq \xi \in \pi_1^{-1}(E) \cap U$ . Now, if  $V$  is a  $\lambda$ -neighborhood of  $\pi_1 \xi$  in  $\vartheta$ , then  $G_0 \cup \pi_1^{-1}(V)$  is a neighborhood of  $\xi$ , so that there is  $\zeta \neq \xi \in \pi_1^{-1}(E) \cap (G_0 \cup \pi_1^{-1}(V))$ . But  $\pi_1^{-1}(E) \cap G_0 = \emptyset$ , and we conclude  $\zeta \in \pi_1^{-1}(V)$ , hence  $\pi_1 \zeta \in E \cap V$ . Further,  $\pi_1$  is injective so  $\pi_1 \zeta \neq \pi_1 \xi$ . Since  $V$  was arbitrary, it follows that  $\pi_1 \xi \in d_\lambda E$ .

Conversely, if  $\pi_1 \xi \in d_\lambda E$ , then every  $\lambda$ -neighborhood  $V$  of  $\pi_1 \xi$  contains some  $\zeta \neq \xi \in E$ ; it follows that  $\pi_1 \xi$  is not isolated and thus for every  $\lambda$ -neighborhood  $U$  of  $\xi$  there is a  $\lambda$ -neighborhood  $V$  of  $\pi_1 \xi$  such that  $\pi_1^{-1}(V) \subseteq U$ . Since  $\pi_1 \xi \in d_\lambda E$  there is some  $\zeta \in E \cap V$  with  $\zeta \neq \xi$  and thus  $\pi_1^{-1}(\zeta) \in U \cap \pi_1^{-1}E$ . Since  $U$  was arbitrary and  $\pi_1^{-1}\zeta \neq \xi$  by the injectivity of  $\pi_1$ , we conclude that  $\xi \in d'_\lambda \pi_1^{-1}E$ .

The proof that  $i_{1+\lambda} \pi_1^{-1}E = \pi_1^{-1}i_{1+\lambda}E$  is analogous but uses Lemma 10.2.2.  $\square$

**Lemma 10.4.** *Suppose  $\mathfrak{X} = \langle [0, \eta], \vec{T}, \mathcal{A} \rangle$  and  $\mathfrak{Y} = \langle [0, \vartheta], \vec{S}, \mathcal{A} \rangle$  are ambiances.*

*Then,  $\mathfrak{X} \otimes_d \mathfrak{Y}$  equipped with  $\mathcal{A} \otimes_d \mathcal{B}$  is also an ambiance.*

*Further, if both ambiances are idyllic, then so is the corresponding product.*

*Proof.* Let  $S = \pi_0^{-1}(S_0) \cup \pi_1^{-1}(S_1) \in \mathcal{A} \otimes_d \mathcal{B}$  and  $\lambda < \Lambda$ . Let us check that  $d_\lambda S \in \mathcal{A} \otimes_d \mathcal{B}$ .

If  $\lambda = 0$  and  $S_0 \neq \emptyset$ , then by Proposition 10.1.5,  $\pi_0^{-1}S_0$  is 0-dense in  $G_1$  and thus  $G_1 \subseteq d_0 S$ .

Meanwhile,  $G_0$  is 0-open and  $\pi_0$  is a  $d$ -map so for  $\xi \in G_0$ ,  $\xi \in d_\lambda S$  if and only if  $\pi_0 \xi \in d_\lambda S_0$ , and we conclude that

$$d_0 S = \pi_0^{-1}d_0 S_0 \cup \pi_1^{-1}[0, \vartheta] \in \mathcal{A} \otimes_d \mathcal{B}.$$

By similar reasoning,

$$i_1 S = \pi_0^{-1}i_1 S_0 \cup \pi_1^{-1}[0, \vartheta] \in \mathcal{A} \otimes_d \mathcal{B},$$

and if the original structures are idyllic this is evidently equal to  $d_0 S$  as well.

Now suppose  $\lambda = 0$  and  $S_0 = \emptyset$ . In this case,  $S = \pi_1^{-1}S_1$  and, by Lemma 10.3,  $d_\lambda S = \pi_1^{-1}d_\lambda S_1 \in \mathcal{A} \otimes_d \mathcal{B}$ .

If the original algebras were idyllic, we note further that

$$d_0 S = \pi_1^{-1}d_0 S_1 = \pi_1^{-1}i_1 S_1 = i_1 S.$$

Finally, if  $\lambda > 0$ , then both projections are  $d$ -maps with respect to the  $\lambda$ -topology and

$$d_\lambda S = d_\lambda \pi_0^{-1} S_0 \cup d_\lambda \pi_1^{-1} S_1 = \pi_0^{-1} d_\lambda S_0 \cup \pi_1^{-1} d_\lambda S_1,$$

with the analogous equalities holding for  $i_{1+\lambda} S$ , from which all required claims follow easily.  $\square$

With this we conclude the topological constructions we shall need. Now, we turn to the last ingredient in the completeness proof: the modal logic J.

## 11 The logic J

As we have seen,  $\text{GLP}_\Lambda$  has no non-trivial Kripke frames. In order to avoid this issue, we pass to a weaker logic, Beklemishev's J. This was introduced in [1] and here we only review the necessary results without proof. For this logic we shall only use modalities  $n < \omega$  and replace Axiom 4 of  $\text{GLP}_\Lambda$  by the two axioms

- 6.  $[n]\phi \rightarrow [m][n]\phi$ , for  $n \leq m$  and
- 7.  $[n]\phi \rightarrow [n][m]\phi$ , for  $n < m$ .

The logic J is sound and complete for the class of finite Kripke models  $\langle W, \langle <_n \rangle_{n < N}, \llbracket \cdot \rrbracket \rangle$  such that

- 1. the relations  $<_n$  are transitive and well-founded,
- 2. if  $n < m$  and  $w <_m v$  then  $<_n(w) = <_n(v)$  and
- 3. if  $n < m$  then  $w <_m v <_n u$  implies that  $w <_n u$ .

Here,  $<_n(w) = \{v : v <_n w\}$ . It will also be convenient to define  $w \ll_n v$  if for some  $m \geq n$ ,  $w \ll_m v$ . Let  $\sim_n$  denote the symmetric, transitive, reflexive closure of  $\ll_n$  and let  $[w]_n$  denote the equivalence class of  $w$  under  $\sim_n$ . Write  $[w]_{n+1} <_n [v]_{n+1}$  if there exist  $w' \in [w]_n, v' \in [v]_n$  such that  $w' <_n v'$ .

Then, say  $W$  is *tree-like* if

- 1. for each  $w \in W$  and  $n \leq N$ ,  $[w]_n / \sim_{n+1}$  is a tree under  $<_n$  and
- 2. if  $[w]_{n+1} <_n [v]_n$  then  $w <_n v$ .

With this we may state the following completeness result from [1]:

**Lemma 11.1.** *Any J-consistent formula can be satisfied on a finite, tree-like J-frame.*

Thus if we can reduce  $\text{GLP}_\Lambda$  to J, we will immediately obtain finite Kripke models. For this, given a formula  $\phi$ , let  $N$  be the largest modality appearing in  $\phi$  and define

$$M(\phi) = \bigwedge_{\substack{[n]\psi \in \text{sub}(\phi) \\ n < m \leq N}} [n]\psi \rightarrow [m]\psi.$$

Then we set  $M^+(\phi) = M(\phi) \wedge \bigwedge_{n \leq N} [n]M(\phi)$ .

The following is also proven in [1]:

**Lemma 11.2.** *For any formula  $\phi \in \mathbf{L}_\omega$ ,  $\mathbf{GLP}_\omega \vdash \phi$  if and only if*

$$\mathbf{J} \vdash M^+(\phi) \rightarrow \phi.$$

To prove completeness, it then suffices to construct a J-model of a given formula and then “pull back” the valuations onto a topological model. Hence it is important to identify the appropriate maps for such pullbacks.

First observe that a partially ordered set  $\langle W, < \rangle$  can be identified with a topological space by letting  $U \subseteq W$  be open if, whenever  $v < w$  and  $w \in U$ , it follows that  $v \in U$ . If  $\mathfrak{W} = \langle W, \langle <_n \rangle_{n \leq N} \rangle$  is a J-frame, we will let  $\mathfrak{W}_n$  be the topological space associated to  $\langle W, <_n \rangle$ . Below, we say  $w \in W$  is a *hereditary*  $n$ -root if  $w$  is  $<_k$ -maximal for all  $k \geq n$ .

**Definition 11.1.** *Let  $\mathfrak{X} = \langle X, \langle \mathcal{T}_n \rangle_{n \leq N} \rangle$  be a polytopological space and  $\mathfrak{W} = \langle W, \langle <_n \rangle_{n \leq N} \rangle$  a J-frame.*

*A function  $f : \mathfrak{X} \rightarrow \mathfrak{W}$  is a J-map if*

1.  $f : \mathfrak{X}_N \rightarrow \mathfrak{W}_N$  is a d-map
2.  $f : \mathfrak{X}_n \rightarrow \mathfrak{W}_n$  is open for  $n < N$
3. if  $n < N$  and  $w$  is a hereditary  $(n+1)$ -root then  $f^{-1}(\ll_n(w))$  and

$$f^{-1}(\ll_n(w) \cup \{w\})$$

*are  $n$ -open*

4. if  $n < N$  and  $w$  is a hereditary  $(n+1)$ -root then  $f^{-1}(w)$  is  $n$ -discrete.

With this we have the following, proven in [5]:

**Lemma 11.3.** *If  $\mathfrak{W} = \langle W, \langle <_n \rangle_{n \leq N}, \ll \cdot \rangle \rangle$  is a J-model such that  $M^+(\phi)$  is valid on  $\mathfrak{W}$ ,  $\mathfrak{X} = \langle X, \langle \mathcal{T}_n \rangle_{n \leq N} \rangle$  is a GLP-space and  $f : \mathfrak{X} \rightarrow \mathfrak{W}$  a J-map, then there is a valuation  $\ll \cdot \rrbracket$  on  $\mathfrak{X}$  such that  $\ll \psi \rrbracket = f^{-1} \ll \psi \rrbracket$  for all  $\psi \in \text{sub}(\phi)$ .*

We conclude with a simple observation, also established in [5]:

**Lemma 11.4.** *Let  $\mathfrak{X}, \mathfrak{Y}$  be polytopological spaces and  $\mathfrak{W}$  a J-frame.*

*Then, if  $f : \mathfrak{X} \rightarrow \mathfrak{Y}$  is a d-map and  $g : \mathfrak{Y} \rightarrow \mathfrak{W}$  is a J-map it follows that  $gf$  is a J-map.*

In the next section we shall exploit the completeness of the logic J for finite frames together with Lemma 11.3 to construct GLP-ambiances satisfying any consistent formula.

## 12 Completeness

Given an increasing sequence of ordinals  $\vec{\lambda} = \langle \lambda_n \rangle_{n \leq N}$  and a polytopological space  $\langle X, \langle \mathcal{T}_\xi \rangle_{\xi < \Lambda} \rangle$ , we will say a map  $f : X \rightarrow W$  is a  $\vec{\lambda}$ -map if it is a J-map on  $\langle X, \langle \mathcal{T}_{\lambda_n} \rangle_{n \leq N} \rangle$ . These maps will allow us to focus on finitely many modalities at one time in the completeness proof.

Say a  $\vec{\lambda}$ -map is *suitable* if it is a surjective function of the form  $f : (\Theta + 1) \rightarrow W$ , for the 0-root  $w_0$  of  $W$  we have  $f^{-1}(w_0) = \{\Theta\}$  and  $\Theta$  lies in the range of  $e$ . Below, if  $\text{hgt}(<_i)$  denotes the *height* of  $<_i$ , that is, the maximal  $k$  such that there exist  $w_0 <_i w_1 <_i \dots <_i w_k$ .

**Lemma 12.1.** *Given a J-frame  $\mathfrak{W} = \langle W, \langle <_i \rangle_{i \leq I} \rangle$  and ordinals  $\vec{\lambda} = \langle \lambda_i \rangle_{i \leq I}$  all less than  $\Lambda$ , there exists an idyllic  $\Lambda$ -ambience  $\mathfrak{X}$  based on  $\Theta < e^{1+\Lambda}1$  and a suitable  $\vec{\lambda}$ -map  $f : \mathfrak{X} \rightarrow \mathfrak{W}$ .*

*Proof.* We proceed as in the proof of an analogous result in [5].

Suppose  $\langle W, \langle <_i \rangle_{i \leq I} \rangle$  is a J-frame and  $\lambda_0, \dots, \lambda_I$  are ordinals. We work by induction on  $I$  with a secondary induction on  $\text{hgt}(<_0)$  to construct a suitable  $\vec{\lambda}$ -map. Without loss of generality, we assume  $\lambda_0 = 0$ , for otherwise we can always let  $<_0 = \emptyset$ .

**Case 1:**  $I = 0$ . In this case it is known that there is an ordinal  $\Theta < \omega^\omega = e(\omega)$  and a suitable  $d$ -map  $g : (\Theta + 1)_1 \rightarrow W$  [6]. Note that since  $\Theta < \omega^\omega$ , it has no points of limit rank and hence the interval topology is already limit-maximal so that  $\mathcal{P}(\Theta + 1)$  is an idyllic algebra.

**Case 2:**  $\text{hgt}(<_0) = 0$ . Here we have that  $<_0 = \emptyset$ . For  $0 < i \leq I$  let  $\lambda'_i = -\lambda_1 + \lambda_i$  and consider the J-frame  $\langle W, \langle <_{i+1} \rangle_{0 \leq i < I} \rangle$ , where  $<_0$  has been removed. By induction on  $I$  we may assume there is an idyllic ambience  $\mathfrak{X}$  based on a polytopology  $\vec{\mathcal{T}}$  on an ordinal  $\Theta + 1 < e^{1+(-\lambda_1+\Lambda)}1$  and a suitable  $\vec{\lambda}'$ -map  $g$  from  $\mathfrak{X}$  onto  $W$ .

Let  $\Omega = e^{1+\lambda_1}\ell\Theta$  and use Lemma 8.4 to construct a BG-polytopology

$$\mathfrak{Y} = \langle \Omega + 1, \langle \mathcal{S}_\lambda \rangle_{\lambda < \lambda_1} \rangle.$$

Note that

$$e^{1+\lambda_1}\ell\Theta < e^{1+\lambda_1+(-\lambda_1+\Lambda)}1 = e^{1+\Lambda}1.$$

Then, by Theorem 9.1 there is a  $\Lambda$ -reductive map

$$f = \mathfrak{d}_{\lambda_1}^{\ell\Theta} : (\Omega + 1)_{1+\lambda_1} \rightarrow (\Theta + 1)_1,$$

so that by Lemma 9.1  $f : \mathfrak{Y}_{\lambda_1}^- \rightarrow \mathfrak{X}_0$  is a  $d$ -map. By Lemma 8.2, there exists a BG-space  $\mathfrak{Y}_{\lambda_1}$  extending  $\mathfrak{Y}_{\lambda_1}^-$  such that  $f : \mathfrak{Y}_{\lambda_1} \rightarrow \mathfrak{X}_0$  is a  $d$ -map.

Hence we may use Lemma 8.5 to define BG-topologies  $\langle \mathcal{S}_\xi \rangle_{\lambda_1 < \xi \leq \lambda_I}$  on  $\Omega + 1$  such that  $f : \mathfrak{Y}_{\lambda_1+\zeta} \rightarrow \mathfrak{X}_\zeta$  is a  $d$ -map for all  $\zeta$ ; in particular, for each  $i \in (0, I]$  we have that  $f : \mathfrak{Y}_{\lambda_i} \rightarrow \mathfrak{X}_{\lambda'_i}$  is a  $d$ -map.

It follows that  $f : \mathfrak{Y} \rightarrow \mathfrak{X}$  is a  $d$ -lift, and by Lemma 10.1,  $f^{-1}\mathcal{A}$  is an idyllic  $d$ -algebra. Further, by Lemma 11.4 we know that  $gf : \mathfrak{Y} \rightarrow W$  is a suitable  $\bar{\lambda}$ -map, as needed.

**Case 3:**  $\text{hgt}(<_0) = m > 0$ . Let  $w$  be the 0-root of  $W$  and  $w_0, \dots, w_N$  be its  $<_0$ -daughters which are hereditary 1-roots.

Let  $V = \{w\} \cup \ll_1(w)$  and  $W_n = \ll_0(w_n)$ . Then we have, as in Case 2, an idyllic ambiance  $\mathfrak{X}$  based on an ordinal  $\vartheta + 1 < e^{1+\Lambda}1$  and by induction on  $I$  a  $\bar{\lambda}$ -map  $f$  from  $\vartheta$  to  $V$ , as well as for each  $n \leq N$  an idyllic ambiance  $\mathfrak{Y}_n$  based on an ordinal  $\eta_n + 1 < e^{1+\Lambda}1$  and a suitable  $\bar{\lambda}$ -map  $f_n$  from  $\eta_n$  onto  $W_n$ .

Let  $\eta = (\eta_0 + 1) + (\eta_1 + 1) + \dots + \eta_N$ ,  $\mathfrak{Y} = \bigoplus_{n \leq N} \mathfrak{Y}_n$  and  $\mathfrak{Z} = \mathfrak{Y} \otimes_d \mathfrak{X}$  with associated maps  $\pi_0$  and  $\pi_1$ . Let  $G_0$  be the domain of  $\pi_0$  and  $G_1$  be its complement. Define  $g : \mathfrak{Y} \rightarrow W$  by

$$g((\eta_0 + 1) + \dots + (\eta_{n-1} + 1) + \xi) = f_n(\xi)$$

and  $h : \mathfrak{Z} \rightarrow W$  by

$$h(\xi) = \begin{cases} g\pi_0(\xi) & \text{if } \xi \in G_0, \\ f\pi_1(\xi) & \text{otherwise.} \end{cases}$$

Note that  $\Lambda > 0$  so that  $e^{1+\Lambda}1 > \omega$  is closed under sums and products and thus  $\eta \otimes_d \vartheta < e^{1+\Lambda}1$ . Further, by Lemma 10.4,  $\mathfrak{Z}$  is an idyllic ambiance.

Now, let us check that  $h$  satisfies the conditions of Definition 11.1.

1. We know that  $\lambda_I > 0$ , since  $I > 0$ . Thus  $G_0, G_1$  are both  $\lambda_I$ -clopen and hence it is enough to check that  $h \upharpoonright G_j : \mathfrak{Z}_{\lambda_i} \rightarrow \langle W, <_I \rangle$  is a  $d$ -map for  $j = 0, 1$ . But this is immediate from the assumption that  $f, g$  are  $d$ -maps, as are the respective projections.

2. Let  $\lambda = \lambda_i$ ,  $U$  be  $\lambda$ -open and  $v \in h(U)$ . Note that

$$h(U) = g\pi_0(G_0 \cap U) \cup f\pi_1(G_1 \cap U).$$

First note that  $G_0 \cap U$  is  $\lambda$ -open since  $G_0$  is 0-open, hence  $g\pi_0(G_0 \cap U)$  is also  $\lambda$ -open given that  $g\pi_0$  is a composition of  $\lambda$ -open maps. Meanwhile, for  $\lambda > 1$ ,  $G_1$  is also  $\lambda$ -open from which it follows that  $f\pi_1(G_1 \cap U)$  is  $\lambda$ -open as well.

It remains to check that  $h(U)$  contains a 0-neighborhood around any  $v \in f\pi_1(G_1 \cap U)$ . So suppose  $u <_0 v$ . Since  $g$  is onto  $\bigcup W_n$ , there is some  $\delta \leq \eta$  such that  $g(\delta) = u$ . By Lemma 10.1.5 and using the fact that  $G_1 \cap U \neq \emptyset$  (otherwise  $v$  would not exist), there is  $\gamma \in U \cap \pi_0^{-1}\delta$ . It follows that  $h(\gamma) = g\pi_0(\gamma) = u$ , as desired.

3. If  $i > 0$  and  $v$  is an  $i + 1$  root the claim follows from the assumption that  $f, g$  were already  $J$ -maps and the respective projections are  $d$ -maps.

Meanwhile, if  $i = 0$  and  $v <_0 w$ , we may use the fact that  $g$  is a  $\langle \lambda_i \rangle_{1 \leq i \leq I}$ -map, since here it follows that  $h^{-1}(\ll_0(v)) \subseteq G_0$  so that

$$h^{-1}(\ll_0(v)) = (g\pi_0)^{-1}(\ll_0(v)).$$

But  $g^{-1}(\ll_0(v))$  is a 0-open subset of  $\eta$ , hence  $(g\pi_0)^{-1}(\ll_0(v))$  is open in  $\mathfrak{J}$ . Similarly for  $\{v\} \cup \ll_0(v)$ .

If  $v \ll_1 w$ , then  $<_0(v) = <_0(w)$ , and thus  $h^{-1}(<_0(v)) = G_0$ . But then we see that

$$h^{-1}(\ll_0(v)) = G_0 \cup \pi_1^{-1}f^{-1}(\ll_0(v));$$

since by assumption  $f$  is a J-map,  $f^{-1}(\ll_0(v))$  is 0-open, and therefore by definition  $G_0 \cup \pi_1^{-1}f^{-1}(\ll_0(v))$  is 0-open in the  $d$ -product  $\mathfrak{J}$ . The argument for  $\{v\} \cup \ll_0(v)$  is analogous.

Finally, note that  $f^{-1}(w) = \{\eta \otimes_d \vartheta\}$  so that  $f^{-1}(\ll_0(w)) = W \setminus \{w\} = [0, \eta \otimes_d \vartheta]$ , which is open, as is  $f^{-1}(\{w\} \cup \ll_0(w)) = [0, \eta \otimes_d \vartheta]$ .

4. For  $w$  we have that  $h^{-1}(w) = \{\eta \otimes_d \vartheta\}$ , which is obviously  $\lambda_i$ -discrete for any  $i$ . If  $v <_0 w$  then  $h^{-1}(v) = (g\pi_0)^{-1}(v)$  which is discrete, as  $g\pi_0$  is a J-map.

If  $v \ll_1 w$  then  $h^{-1}(v) = (f\pi_1)^{-1}(v)$ , which similarly must be discrete.  $\square$

With this, we are ready to state and prove our main result.

**Theorem 12.1.** *GLP $_{\Lambda}$  is complete for the class of idyllic ambiances based on some  $\Theta < e^{1+\Lambda}1$ .*

*Proof.* Suppose that  $\phi$  is consistent over GLP $_{\Lambda}$ . Then, by Lemma 2.1,  $\phi^c$  is consistent over GLP $_I$ . By Lemma 11.2,  $M^+(\phi) \wedge \phi$  is consistent over J, and thus by Lemma 11.1, we have a tree-like J-model  $\langle W, \langle <_n \rangle_{n \leq N}, \ll \cdot \rangle$  satisfying  $M^+(\phi) \wedge \phi$ . We may then use Lemma 12.1 to find an idyllic ambiance  $\mathfrak{X}$  based on an ordinal  $\Theta < e^{\Lambda}\omega$  and a surjective  $\vec{\lambda}$ -map  $f : \mathfrak{X} \rightarrow W$ .

Then, by Lemma 11.3, there is a valuation on  $\Theta$  agreeing with  $f^{-1} \ll \cdot \rrbracket$  on  $\text{sub}(\phi)$ . Since  $f$  is surjective,  $f^{-1} \ll \phi \rrbracket \neq \emptyset$ , and thus  $\phi$  is satisfied on  $\mathfrak{X}$ , as desired.  $\square$

As corollaries we get a sequence of completeness results:

**Corollary 12.1.** *GLP $_{\Lambda}$  is complete for both the class of BG-spaces and the class of shifted Icard ambiances based on  $e^{1+\Lambda}1$ .*

*Further, the variable-free fragment GLP $_{\Lambda}^0$  is complete for the class of simple Icard ambiances based on  $e^{\Lambda}1$ .*

*Proof.* Completeness for BG-spaces and shifted Icard ambiances is immediate from Theorem 12.1, as idyllic ambiances may be seen as either kind of structure.

Meanwhile, given any Icard ambiance based on an algebra  $\mathcal{A}$  and satisfying a closed formula  $\phi$ , we use Lemma 7.2 to note that all valuations of  $\phi$  and its subformulas are simple, and hence we obtain a simple ambiance satisfying  $\phi$  by replacing  $\mathcal{A}$  by the class of simple sets.



This gives us an ambience on  $\widehat{\mathfrak{Ic}}_{\Lambda}^{e^{1+\Lambda}1}$ , the shifted Icard space. To pass to an ambience based on  $\mathfrak{Ic}_{\Lambda}^{e^{\Lambda}1}$ , note that  $\ell : \widehat{\mathfrak{Ic}}_{\Lambda}^{e^{1+\Lambda}1} \rightarrow \mathfrak{Ic}_{\Lambda}^{e^{\Lambda}1}$  is a  $d$ -map and thus by an easy induction preserves valuations of closed formulas.  $\square$

The completeness result for simple ambiances is not new, as it was already proven by Icard for  $\text{GLP}_{\omega}$  in [11] and by Joosten and I for arbitrary  $\text{GLP}_{\Lambda}$  in [9]. However, the current argument is quite different from those used in previous works.

### 13 Worms and the lower bound

As it turns out, our bound of  $e^{1+\Lambda}1$  is sharp. To show this, let us consider *worms*.

A worm is a formula of the form

$$\langle \lambda_0 \rangle \dots \langle \lambda_I \rangle \top.$$

These formulas correspond to iterated consistency statements, and indeed can be used to study the proof-theoretic strength of many theories related to Peano Arithmetic, as Beklemishev has shown [2].

Worms are well-ordered by their *consistency strength*. Let us denote the set of worms with entries less than  $\Lambda$  by  $\mathbb{W}^{\Lambda}$ ; then, given worms  $\mathfrak{v}, \mathfrak{w} \in \mathbb{W}^{\Lambda}$ , write  $\mathfrak{v} \triangleleft \mathfrak{w}$  if  $\text{GLP}_{\Lambda} \vdash \mathfrak{w} \rightarrow \Diamond \mathfrak{v}$ .

The relation  $\triangleleft$  we have just defined is a well-order [2, 10]. Thus we may compute the order-type of a worm  $\mathfrak{w} \in \mathbb{W}^{\Lambda}$ :

$$o(\mathfrak{w}) = \sup_{\mathfrak{v} \triangleleft \mathfrak{w}} (o(\mathfrak{v}) + 1).$$

**Lemma 13.1.** *Let  $\mathfrak{w}$  be a worm,  $\mathfrak{X} = \langle X, \langle \mathcal{T}_{\lambda} \rangle_{\lambda < \Lambda}, \llbracket \cdot \rrbracket \rangle$  be a GLP-model and  $x \in X$ .*

*Then, if  $x \in \llbracket \mathfrak{w} \rrbracket$ , it follows that  $\rho(x) \geq o(\mathfrak{w})$ .*

*Proof.* By induction on  $o(\mathfrak{w})$ . For the base case, note that if  $o(\mathfrak{w}) = 0$  then we vacuously have that  $x \in \llbracket \mathfrak{w} \rrbracket$  implies  $\rho(x) \geq 0$ .

For the inductive step, if  $\mathfrak{v} \triangleleft \mathfrak{w}$  and  $U$  is any neighborhood of  $x$ , since  $\vdash \mathfrak{w} \rightarrow \Diamond \mathfrak{v}$ , it follows that there is  $y \in U$  satisfying  $\mathfrak{v}$ . By induction on  $\mathfrak{v} \triangleleft \mathfrak{w}$ , we have that  $\rho(y) \geq o(\mathfrak{v})$ .

We then see that

$$\rho(x) \geq \sup_{\mathfrak{v} \triangleleft \mathfrak{w}} (o(\mathfrak{v}) + 1) = o(\mathfrak{w}).$$

$\square$

It will be convenient to review the calculus for computing  $o$  that is given in [10]. First, if  $\mathfrak{v} = \langle \xi_1 \rangle \dots \langle \xi_N \rangle \top$  and  $\mathfrak{w} = \langle \zeta_1 \rangle \dots \langle \zeta_M \rangle \top$ , define

$$\mathfrak{v}0\mathfrak{w} = \langle \xi_1 \rangle \dots \langle \xi_N \rangle \langle 0 \rangle \langle \zeta_1 \rangle \dots \langle \zeta_M \rangle \top.$$

Further, if  $\alpha$  is any ordinal, set

$$\alpha \uparrow \mathfrak{w} = \langle \alpha + \zeta_1 \rangle \dots \langle \alpha + \zeta_M \rangle \top.$$

**Lemma 13.1.** *Let  $\mathfrak{v}, \mathfrak{w}$  be worms and  $\alpha$  an ordinal.*

*Then,*

$$o(\top) = 0 \tag{3}$$

$$o(\mathfrak{v}0\mathfrak{w}) = o(\mathfrak{w}) + 1 + o(\mathfrak{v}) \tag{4}$$

$$o(\alpha \uparrow \mathfrak{w}) = e^\alpha o(\mathfrak{w}). \tag{5}$$

**Theorem 13.1.**  *$\text{GLP}_\Lambda$  is incomplete for the class of BG-spaces or Icard ambiances based on any fixed  $\Theta < e^{1+\Lambda}1$ .*

*Proof.* Note that  $\ell(e^{1+\Lambda}1) = e^\Lambda 1$ . Now, if  $\text{GLP}_\Lambda$  is complete for the class of models based on  $\Theta$ , in particular any worm  $\mathfrak{w} \in \mathbb{W}^\Lambda$  must be satisfiable on one such model  $\mathfrak{X}$ , which implies by Lemma 13.1 that there must be  $\vartheta \in \Theta$  with  $\ell(\vartheta) \geq o(\mathfrak{w})$ . Thus it suffices to show that

$$\sup_{\mathfrak{w} \in \mathbb{W}^\Lambda} o(\mathfrak{w}) \geq e^\Lambda 1.$$

To do this, first assume  $\Lambda = \lambda + 1$ . Then we have that

$$o(\langle \lambda \rangle^n \top) = e^\lambda o(\langle 0 \rangle^n \top) = e^\lambda n.$$

The last equality is obtained by repeated applications of (4).

But then, by Lemma 4.1.1 we see that

$$\rho(\mathfrak{X}) \geq \lim_{n \rightarrow \omega} e^\lambda n = e^{\lambda+1} 1.$$

Meanwhile, if  $\Lambda \in \text{Lim}$ ,

$$\rho(\mathfrak{X}) \geq \sup_{\lambda < \Lambda} o(\langle \lambda \rangle \top) = \sup_{\lambda < \Lambda} e^\lambda 1 = e^\Lambda 1.$$

In either case  $\rho(\mathfrak{X}) = \sup_{\vartheta < \Theta} \ell \vartheta \geq e^\Lambda 1$ , from which it follows by Lemma 4.1 that  $\Theta \geq e^{1+\Lambda} 1$ .  $\square$

## 14 Concluding remarks

The goal of this paper was essentially to answer two main questions. The first is perhaps not so much *Is  $\text{GLP}_\Lambda$  complete for its topological semantics independently of  $\Lambda$ ?* as, rather, *What is needed to construct topological models of  $\text{GLP}_\Lambda$ ?* For this we had to introduce several tools that were not required in the case  $\Lambda = \omega$ . Most notable is the use of hyperlogarithms and -exponents, already employed in [9] to study models of the closed fragment, and the addition of new  $d$ -maps to our toolkit. Aside from possible connections to proof theory, these

are novel constructions in scattered topology and might spark some independent interest.

The second question is, *Are there good constructive semantics for  $\text{GLP}_\Lambda$ ?* Icard ambiances are a possible answer to this question. Not only are the topologies easily definable, unlike the non-constructive BG-topologies, but if one analyzes the proof of Lemma 12.1, all sets that appear in valuations are constructive as well. As such, Icard ambiances may be well-suited for applications in the proof theory of systems much stronger than Peano Arithmetic – perhaps the ultimate motivation for contemporary work in provability logic.

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